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A SMALL-GAIN THEOREM FOR RANDOM DYNAMICAL SYSTEMS WITH INPUTS AND OUTPUTS*

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Abstract. A formalism for the study of random dynamical systems with inputs and outputs (RDSIO) is introduced. An axiomatic framework and basic properties of RDSIO are developed, and a theorem is shown that guarantees the stability of interconnected systems.

Key words. random dynamical systems, control systems, small-gain theorems

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1. Introduction.

For deterministic systems, there is a well-developed and constructive theory of systems interconnections and feedback, such as the very successful and widely applied backstepping approach [18, 16]. Thus, it is natural to attempt to extend such work to stochastic systems. Indeed, much excellent research has been done pursuing such extensions, notably studies on stochastic stabilization [13, 27], as well as feedback stabilization using noise to state stability [10, 9, 33]. In this paper we pursue a different approach, based instead upon Ludwig Arnold's notion of random dynamical systems (RDS), which provides an elegant and deep axiomatization of random dynamics.

An RDS is made up of two ingredients, a stochastic but autonomous noise process described by a measure-preserving dynamical system (MPDS), combined with a classical dynamical system that is driven by this process. The noise process is used to encapsulate randomly fluctuating parameters that may arise from environmental perturbations, measurement errors, or internal variability. The RDS formalism provides a seamless integration of classical ergodic theory with modern dynamical systems, giving a theoretical framework parallel to classical smooth and topological dynamics (stability, attractors, bifurcation theory) while allowing one to treat in a unified way the most important classes of dynamical systems with randomness, such as random differential or difference equations (basically, deterministic systems with randomly changing parameters) or stochastic ordinary and partial differential equations (white noise or, more generally, semimartingale-driven systems as studied in the Itô calculus). We refer to [2] for a textbook presentation. The RDS formalism takes full advantage of the power of ergodic theory. As a simple illustration, suppose that we want to study the scalar affine system $\dot{x} = ax$, where a is randomly varying, $a = a(\omega)$. If it were the case that all realizations of the parameter a are uniformly bounded away from the origin, $a(\omega) \leq -\lambda < 0$ for all ω, then stability would not be an issue. However, it may be the case that $\mathbb{E}[a] < 0$, even though a may still take nonnegative values with positive probability. In this case, almost-sure convergence to zero will follow from

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the fundamental (and nontrivial) multiplicative ergodic theorem [2, Chapter 3], the random proxy for linear algebra upon which much of RDS theory relies. (See also the discrete-time example in subsection 3.1, Example 3.15, and Remark 3.16 below.)

If we now add an external input or control u and consider the forced system $\dot{x} = ax + bu$, we similarly have an input-to-state stability property, yielding a globally attracting state $\mathcal{K}(u)$ for each input u, provided that $\mathbb{E}[a] < 0$. (Refer to Example 3.15 for the details.) This paper systematically develops such an extension of RDS to encompass inputs and outputs, a notion which we term *random dynamical systems with inputs and outputs (RDSIO)*. We sketched this study (with no outputs) in the conference paper [22] and proved a basic convergent-input-to-convergent-state (CICS) theorem in the book chapter [23]. A major contribution of this project is the precise formulation of the way in which the inputs are shifted in the semigroup (cocycle) property, and the focus on stochastic inputs, which is essential in order to develop a theory of interconnected subsystems, as an input to one system in such an interconnection is typically obtained by combining the (necessarily random) outputs of other subsystems.

CICS theorems provide conditions under which convergence of the input implies convergence of the state process (for given random initial conditions). Observe that, even for deterministic systems that are globally asymptotically stable with respect to constant inputs, the CICS property may not be observed. This led to the introduction of the notions of input-to-state stability [30] and of monotone systems with inputs [1] for deterministic systems, either of which allows one to obtain CICS results. In [23] and, again, in this paper, we pursue a monotone systems approach, expanding upon the useful framework recently developed by Chueshov [6] for monotone RDS (without inputs). After considerably refining the basic concepts from [23] and proving additional basic results, we turn to our main new contribution, the formulation and proof of a "random small-gain" theorem that guarantees global convergence (to a unique equilibrium) of interconnected systems.

Organization of the paper. In the rest of this introduction, we discuss a simple biochemical circuit that will help illustrate our main result. Then, in section 2, we review the concept of MPDs, introducing some notation and terminology not found in [2, 6] to facilitate the discussion. Relevant growth conditions and the mode of convergence with respect to which we study asymptotic behavior of stochastic processes in this work are then described in detail. RDSIO are introduced in section 3. A more general version of the CICS result for monotone random dynamical systems with inputs (RDSI) from [23] is presented, and a thorough treatment of output functions is given. In section 4, we present and prove our (random) small-gain theorem, giving several examples illustrating how it can be used to establish global convergence to a unique equilibrium for RDS. Some results needed in order to verify the small-gain property for these examples appeal to the theory of Thompson metrics and are outlined in section 5. Section 6 briefly discusses some possible future directions. Some technical details on the spaces that we consider are collected in Appendix A.

The reader familiar with stochastic differential equations (SDE), particularly in the context of RDS, will certainly note their omission among the examples treated in this paper. This difficult choice was made for pedagogical reasons. On the one hand, a complete treatment of SDE, including perfection of cocycles, would have significantly increased the length and technical complexity of this paper. On the other hand, differential and difference equations are natural continuous- and discrete-time analogues of one another, and for which several examples can be explicitly calculated. This makes them the most natural prototypes to motivate and illustrate the theory.

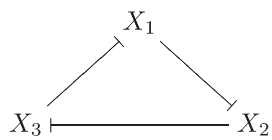

FIG. 1. *Biochemical circuit. The symbol "$X \dashv Y$" means that species X represses the production of species Y.*

Application of the small-gain theorem to the stabilization of closed systems: A motivating example. We introduce here a simple biochemical circuit that will help illustrate our main result. The system involves three chemical species X_1, X_2, and X_3 that interact with one another as illustrated in Figure 1. Systems of this type are routinely studied as biological models of gene repression, and a synthetic construct was implemented as the repressilator circuit in [11], using the genes *lacI*, *tetR*, and *cI*. The simplest mathematical model, when only protein products are used to represent the species (thus omitting intermediate mRNA, posttranslational modifications, and so forth) uses three time-dependent real variables ξ_1, ξ_2, and ξ_3 to denote the concentrations of X_1, X_2, and X_3, respectively, and results in a system of differential equations as follows:

$$\dot{\xi}_i = a_i \xi_i + h_i(\xi_{i-1}) \,, \quad i = 1, 2, 3 \,,$$

with indices taken modulo three, so $\xi_0 = \xi_3$. Here, $a_1, a_2, a_3 < 0$ are interpreted as rates of degradation, and h_1, h_2, h_3 are nonincreasing functions of their arguments, modeling the repression mechanism. Our considerations apply equally well to variants such as the Goodwin model of gene expression and other models that are standard in molecular biology [24, 14].

It is natural that the rates of degradation, as well as the strength of the interactions between the species, may depend on environmental factors such as temperature or pressure, as well as the concentrations of other biochemical compounds not explicitly modeled. Furthermore, this dependence may be intrinsically noisy. If this is the case, then a more realistic model would be a system of differential equations of the form

$$\dot{\xi}_i = a_i(\lambda_t(\omega))\xi_i + h_i(\lambda_t(\omega), \xi_{i-1}) \,, \quad i = 1, 2, 3 \,, \tag{1.1}$$

where $(\lambda_t)_{t \geqslant 0}$ is a stochastic process evolving on a parameter space Λ encoding all relevant internal and external random factors upon which the dynamics of the circuit depends. This yields a system which is now effectively parametrized by a random outcome ω.

When $(\lambda_t)_{t \geqslant 0}$ is stationary, in other words, when

$$\mathbb{P}(\lambda_{t_1+s} \in A_1, \ldots, \lambda_{t_k+s} \in A_k) \equiv \mathbb{P}(\lambda_{t_1} \in A_1, \ldots, \lambda_{t_k} \in A_k) \,,$$

system (1.1) can be canonically rewritten as a *random differential equation* (*RDE*)

$$\dot{\xi}_i = a_i(\theta_t \omega)\xi_i + h_i(\theta_t \omega, \xi_{i-1}) \,, \quad i = 1, 2, 3 \,, \tag{1.2}$$

$$\xrightarrow{u_1} X_1 \xrightarrow{h_1(\xi_1)} \qquad \xrightarrow{u_2} X_2 \xrightarrow{h_2(\xi_2)} \qquad \xrightarrow{u_3} X_3 \xrightarrow{h_3(\xi_3)}$$

FIG. 2. *Decomposition of the biochemical circuit from Figure* 1 *into input/output modules. In each panel,* u_i *indicates the input into the node* X_i *and* $h_i(\xi_i)$ *indicates the ensuing readout—a function of the current state of the system.*

driven by a measure-preserving group action $\theta\colon \mathbb{R}_{\geqslant 0} \times \Omega \to \Omega$ on the probability space induced on $\Lambda^{\mathbb{R}_{\geqslant 0}}$ by $(\lambda_t)_{t\geqslant 0}$. Under suitable hypotheses, (1.2) generates an RDS in the sense of Arnold [2]. This is the framework within which we shall analyze systems such as (1.1). More specifically, we will be concerned with global convergence to a unique equilibrium, in a sense we shall make precise further down.

The nonlinearity implied in the h_i's makes this system difficult to study directly. Since (1.2) is not cooperative with respect to any orthant cone-induced partial order, one cannot directly analyze (1.2) using global convergence results from the theory of monotone RDS [6, 5]. To overcome these difficulties, we present a decomposition-based alternative inspired on the works of Angeli, Enciso, and Sontag on deterministic systems in [1, 12]. The idea is to look at (1.2) as a network of smaller input/output modules (Figure 2), hopefully easier to analyze, and then derive properties of the closed system from emerging properties of these smaller modules.

The first step is to open up the feedback loop, rewriting the model as a system of *random differential equations with inputs* (*RDEI*)

$$\dot{\xi}_i = a_i(\theta_t\omega)\xi_i + u_t^{(i)}(\omega)\,, \quad i = 1, 2, 3\,, \tag{1.3}$$

together with a set of outputs

$$y_t^{(i)}(\omega) = u_t^{(i+1)}(\omega) = h_i(\theta_t\omega, \xi_{i-1})\,, \quad i = 1, 2, 3\,. \tag{1.4}$$

Observe that (1.3) is now linear and monotone (with respect to the positive orthant cone-induced partial orders) on both state and input variables and therefore much easier to study. In fact, in this particular example, if the group action θ is ergodic and the degradation rates a are negative on average, then one can show that (1.3) has a unique, globally attracting equilibrium $\mathcal{K}(u)$ for each stationary input u. We call the map $\mathcal{K}$ so defined the *input-to-state characteristic (I/S characteristic)* of the system.

Once it has been established that the open-loop system is monotone, satisfies certain growth conditions, and possesses a sufficiently regular I/S characteristic, the next step is to look at the gain of the system. The output function is read at $\mathcal{K}(u)$ for each stationary input u, and an operator $\mathcal{K}^Y$ is so defined on the space of stationary inputs. If this operator has a unique, globally attracting fixed point, then the input/output system (1.3)–(1.4) is said to satisfy the *small-gain condition.* One way to interpret this is to say that the procedure of successfully feeding an input into the system, waiting for a while, reading out the state of the system, then feeding it back into the system, and so on, does not lead to blow ups if the initial input does not blow up. It is not hard to believe that the closed-loop system should have equilibria under such circumstances. Monotonicity assumptions will further constrain the behavior of the system, and it will be possible to show that it has, in fact, a unique, globally attracting equilibrium.

2. Asymptotic behavior of MPDS-driven stochastic processes. In this section, we review the concept of MPDS, originally introduced by Arnold and studied in [2]. Without making stronger assumptions, we supplement Arnold's framework with several new pieces of notation and concepts. In particular, we carefully define growth conditions and modes of convergence which assimilate the already established concept of temperedness for random variables in the context of measure-preserving dynamical systems [6]. These will greatly facilitate our later study of the asymptotic behavior of RDSIO.

2.1. Measure-preserving dynamical systems. Whenever X is a topological space, we use the notation $\mathcal{B}(X)$ for the σ-algebra of Borel subsets of X. An MPDS (also called a "metric dynamical system" in the RDS literature) is an ordered quadruple

$$\theta = (\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathcal{T}})$$

consisting of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a directed topological group $(\mathcal{T}, +, \leqslant)$, and a *measurable flow* $(\theta_t)_{t\in\mathcal{T}}$ of invertible, measure-preserving maps $\Omega \to \Omega$. That is,

$$\theta\colon \mathcal{T} \times \Omega \longrightarrow \Omega$$

is a $(\mathcal{B}(\mathcal{T}) \otimes \mathcal{F})$-measurable group action (meaning that $\theta_{t+s}\omega = \theta_t\theta_s\omega$ for all $s, t \in \mathcal{T}$ and $\omega \in \Omega$) with the property that $\mathbb{P} \circ \theta_t = \mathbb{P}$ for each $t \in \mathcal{T}$. In this context, a set $B \in \mathcal{F}$ is said to be θ*-invariant* if $\theta_t(B) = B$ for all $t \in \mathcal{T}$. The MPDS is said to be *ergodic* if, given $B \in \mathcal{F}$, $\theta_t B = B$ for all $t \in \mathcal{T}$ implies $\mathbb{P}(B) = 0$ or $\mathbb{P}(B) = 1$.

It is often the case that a condition depending on $\omega \in \Omega$ is stated to be satisfied for all $\omega \in \widetilde{\Omega}$, for some θ-invariant $\widetilde{\Omega} \subseteq \Omega$ of full measure (that is, $\theta_t\widetilde{\Omega} = \widetilde{\Omega}$ for all $t \in \mathcal{T}$, and $\mathbb{P}(\widetilde{\Omega}) = 1$), while the subset $\widetilde{\Omega}$ itself need not be specified. Whenever this is the case, we shall say simply "for θ-almost all $\omega \in \Omega$" or write

$$\widetilde{\forall}\omega \in \Omega$$

to mean "for all $\omega \in \widetilde{\Omega}$, for some θ-invariant $\widetilde{\Omega} \subseteq \Omega$ of full measure."

Let X be a topological space, and consider the measurable space $(X, \mathcal{B}(X))$. In the context of RDS, the analogue of a point in the state space X for a deterministic system is a *random variable* $\Omega \to X$, that is, a Borel-measurable map $\Omega \to X$. In this work we use the terms "random variable" and "Borel-measurable map" interchangeably. We denote the family of all random variables $\Omega \to X$ by $X^{\Omega}_{\mathcal{B}}$.

Denote $\mathcal{T}_{\geqslant 0} := \{t \in \mathcal{T}\,;\ t \geqslant 0\}$. Now the analogue of deterministic paths $\mathcal{T}_{\geqslant 0} \to X$ will be θ*-stochastic processes* $\mathcal{T}_{\geqslant 0} \times \Omega \to X$; in other words, $(\mathcal{B}(\mathcal{T}_{\geqslant 0}) \otimes \mathcal{F})$-measurable maps $\mathcal{T}_{\geqslant 0} \times \Omega \to X$. Given any such map q, we write

$$q_t := q(t, \cdot)\colon \Omega \to X\,, \quad t \geqslant 0\,.$$

In particular, $q_t \in X^{\Omega}_{\mathcal{B}}$ for every $t \geqslant 0$. In this work we use the terms "θ-stochastic process" and "trajectory" interchangeably. The family of all θ-stochastic processes $\mathcal{T}_{\geqslant 0} \times \Omega \to X$ shall be denoted by $\mathcal{S}^X_\theta$. Of course, a θ-stochastic process is a stochastic process in the traditional sense; we use the prefix "θ-" only to emphasize the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and directed topological time semigroup $\mathcal{T}_{\geqslant 0}$ specified by the given MPDS.

We identify random variables and θ-stochastic processes which agree θ-almost everywhere. More precisely, given $a, b \in X^{\Omega}_{\mathcal{B}}$, we will often abuse notation and write $a = b$ as long as $a(\omega) = b(\omega)$ for θ-almost every $\omega \in \Omega$. Similarly, given $q, r \in \mathcal{S}^X_\theta$, we shall write $q = r$ so long as $q_t(\omega) = r_t(\omega)$ for every $t \geqslant 0$, for θ-almost every $\omega \in \Omega$.

We discuss next an analogue, in the stochastic setting, of *constant* deterministic paths. For each $s \in \mathcal{T}_{\geqslant 0}$, set

$$\rho_s : \mathcal{S}_\theta^X \longrightarrow \mathcal{S}_\theta^X : q \longmapsto \rho_s(q) \tag{2.1}$$

by

$$[\rho_s(q)]_t(\omega) := q_{t+s}(\theta_{-s}\omega), \quad (t,\omega) \in \mathcal{T}_{\geqslant 0} \times \Omega . \tag{2.2}$$

A θ-stochastic process $\bar{q} \in \mathcal{S}_\theta^X$ is said to be *θ-stationary* if

$$\rho_s(\bar{q}) = \bar{q} \quad \forall s \geqslant 0 ,$$

in the sense of the θ-almost everywhere identification describe above, but with the same exceptional θ-invariant subset of probability zero of Ω for each $s \geqslant 0$—that is,

$$[\rho_s(\bar{q})]_t(\omega) = \bar{q}_t(\omega) \quad \widetilde{\forall}\omega \in \Omega , \quad \forall t \geqslant 0 , \quad \forall s \geqslant 0 .$$

We showed in [23] that a θ-stochastic process $\bar{q} \in \mathcal{S}_\theta^X$ is θ-stationary if and only if there exists a random variable $q \in X_{\mathcal{B}}^\Omega$ such that

$$\bar{q}_t(\omega) = q(\theta_t\omega) \quad \widetilde{\forall}\omega \in \Omega , \quad \forall t \geqslant 0 . \tag{2.3}$$

In this case we say that $\bar{q}$ is *generated by* q. Observe that the *generator* q is uniquely determined, up to a θ-invariant set of measure zero, by

$$q(\omega) = \bar{q}_0(\omega) \quad \widetilde{\forall}\omega \in \Omega .$$

We shall always use an overbar to denote the θ-stationary θ-stochastic process $\bar{q}$ generated by a given random variable q. Observe that θ-stationary θ-stochastic processes reduce to *constant* paths in case Ω is a singleton.

Finally, we discuss how θ-stochastic processes may be *concatenated*. For each $s \geqslant 0$, we define an operator $\Diamond_s \colon \mathcal{S}_\theta^X \times \mathcal{S}_\theta^X \to \mathcal{S}_\theta^X$ as follows. Given any $\xi, \zeta \in \mathcal{S}_\theta^X$, the trajectory $\xi \Diamond_s \zeta$ shall consist of the truncation of ξ at time s, "continued" by ζ from then onward. Since ζ starts to run s units of time later, the ω-argument must be shifted accordingly. More precisely, we define $\xi \Diamond_s \zeta \colon \mathcal{T}_{\geqslant 0} \times \Omega \to X$ by

$$(\xi \Diamond_s \zeta)_t(\omega) = \begin{cases} \xi_t(\omega) , & 0 \leqslant t < s, \\ \zeta_{t-s}(\theta_s\omega) , & s \leqslant t, \end{cases} \quad (t,\omega) \in \mathcal{T}_{\geqslant 0} \times \Omega .$$

When Ω is a singleton, this construction reduces to the standard deterministic way of concatenating paths.

2.2. Precompact trajectories. Throughout this work, we will refer for simplicity to a Banach space which is partially ordered by a solid, normal, minihedral cone simply as a *BMNSO space*. See Appendix A for precise definitions and elementary properties. Unless otherwise specified, we shall assume X and U to be closed order-intervals of separable BMNSO spaces, though not necessarily the same underlying space for both X and U. Typical examples would be closed rectangles in $\mathbb{R}^n$ (such as an orthant or a product of finite intervals), with the order associated to the nonnegative orthant cone, which induces in it the *northeast* partial order in which vectors are compared coordinatewise. BMNSO spaces are the setting of the main result in this work, namely, Theorem 4.4, the small-gain theorem for RDS. The assumption that X

and U be closed order-intervals guarantees that infima and suprema of subsets of X or U, when they exist, also belong to the set. Much of what will be discussed would still make sense in a more general setting, but this assumption greatly simplifies the presentation.

A much more subtle assumption we shall tacitly make is that the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ constituting the underlying MPDS is *complete*; that is, every subset of a set of probability zero is measurable. This assumption will simplify the presentation with regards to measurability matters, as noted in Remark 2.4 following Definition 2.3.

Finally, we shall assume that the underlying directed topological group $\mathcal{T}$ is always either $\mathbb{Z}$ or $\mathbb{R}$. In particular, $\mathcal{T}_{\geqslant 0}$ always contains a sequence going to infinity, which shall also come in handy in establishing measurability properties.

Recall that the *pullback* of a θ-stochastic process $\xi \in \mathcal{S}_\theta^X$ is the θ-stochastic process $\check{\xi} \in \mathcal{S}_\theta^X$ defined by

$$\check{\xi}_t(\omega) := \xi_t(\theta_{-t}\omega)\,, \quad (t,\omega) \in \mathcal{T}_{\geqslant 0} \times \Omega\,.$$

In this work asymptotic behavior will be considered in the pullback sense. We will always use the check mark ˇ to indicate the pullback of the θ-stochastic process being accented.

We introduce a few more concepts pertaining to the asymptotic behavior of θ-stochastic processes.

Definition 2.1 (tails). *The* tail *(*from moment τ*) of the pullback trajectories of a θ-stochastic process $\xi \in \mathcal{S}_\theta^X$ is the set-valued function $\beta_\xi^\tau : \Omega \to 2^X \backslash \{\varnothing\}$ defined by*

$$\beta_\xi^\tau(\omega) := \{\xi_t(\theta_{-t}\omega)\,;\ t \geqslant \tau\}\,, \quad \omega \in \Omega\,,$$

for each $\tau \geqslant 0$.

Definition 2.2 (precompact trajectories). *A θ-stochastic process $\xi \in \mathcal{S}_\theta^X$ is said to be* precompact *if $\beta_\xi^0(\omega)$ is precompact for θ-almost all $\omega \in \Omega$. We shall denote the family of all precompact θ-stochastic processes $\mathcal{T}_{\geqslant 0} \times \Omega \to X$ by $\mathcal{K}_\theta^X$.*

Note that $\beta_\xi^\tau(\omega) \subseteq \beta_\xi^\sigma(\omega)$ whenever $\tau \geqslant \sigma \geqslant 0$. So, from this definition it follows that also $\beta_\xi^\tau(\omega)$ is precompact for every $\tau \geqslant 0$, for θ-almost every $\omega \in \Omega$.

Let us further motivate the definitions just introduced. When $\Omega = \{\omega\}$ is a singleton—that is, in the deterministic case—the tail from moment τ of the pullback trajectory of a θ-stochastic process $\xi \in \mathcal{S}_\theta^X$ reduces to the image from $t = \tau$ onward of the given (deterministic) path. So, Definition 2.1 generalizes this concept from the deterministic theory. Definition 2.2 then generalizes the property that the image of a deterministic path is precompact, asking that this be true θ-almost surely in the stochastic scenario.

Precompact θ-stochastic processes evolving on a separable BMNSO space are particularly well-behaved. In this section we shall develop notions of lim inf and lim sup for such processes. These concepts will be the backbone of the constructions leading up to Theorems 3.13 (random CICS) and 4.4 (small-gain theorem).

It follows from Proposition A.4 that the infima and suprema in the definition below are well-defined random variables. It follows from the tacit assumption that X is a closed order-interval that they belong to X. Measurability is a more complicated issue which we discuss in the remark right after the definition.

Definition 2.3 (lower and upper tails). *Given a precompact trajectory $\xi \in \mathcal{K}_\theta^X$, the net $(a_\tau)_{\tau \geqslant 0}$ of random variables $\Omega \to X$ defined by*

$$a_\tau(\omega) := \inf \beta_\xi^\tau(\omega) = \inf_{t \geqslant \tau} \xi_t(\theta_{-t}\omega)\,, \quad \omega \in \Omega\,, \quad \tau \geqslant 0\,,$$

is referred to as the lower tail *(*of the pullback trajectories*) of* ξ*. Similarly, the net* $(b_\tau)_{\tau\geqslant 0}$ *of random variables* $\Omega \to X$ *defined by*

$$b_\tau(\omega) := \sup \beta_\xi^\tau(\omega) = \sup_{t\geqslant\tau} \xi_t(\theta_{-t}\omega)\,, \quad \omega\in\Omega\,, \quad \tau\geqslant 0\,,$$

is referred to as the upper tail *(*of the pullback trajectories*) of* ξ*.*

Remark 2.4. The measurability of the a_τ's and b_τ's in the above definition is a subtle issue. We shall not discuss it here in too much detail lest it distract us from our primary objective in this work—the small-gain theorem. We do, however, briefly describe how it can be settled with well-established results and techniques from the theory of random sets, then refer the reader to [21] for a more thorough account. After a simple reduction [15, Proposition 1.4, p. 142], one may apply the measurable projection theorem [8, Proposition 8.4.4, p. 281] to show that the tails β_ξ^τ of the pullback trajectories of ξ are random sets. (Recall that, given a topological space X, a multifunction $D\colon \Omega\to 2^X$ is said to be a *random set* (or *measurable*) if $M^{-1}(U) := \{\omega\in\Omega\,;\ D(\omega)\cap U\neq\varnothing\}$ is $\mathcal{F}$-measurable for every open $U\subseteq X$.) This is where the assumption that the underlying space X is separable is needed. The argument goes along the lines of the proof of [6, Proposition 1.5.1, pp. 32–33]. This is where the assumption that $(\Omega,\mathcal{F},\mathbb{P})$ is complete comes in; otherwise we can only guarantee measurability with respect to the σ-algebra of "universally measurable" subsets of Ω associated with the underlying measurable space $(\Omega,\mathcal{F})$—which may be larger than $\mathcal{F}$. It then follows from Proposition A.4 and [6, Theorem 3.2.1, p. 90] that a_τ and b_τ are measurable random variables for each $\tau\geqslant 0$.

Since $\beta_\xi^\tau\subseteq\beta_\xi^\sigma$ whenever $\tau\geqslant\sigma\geqslant 0$, it follows straight from the definition of infima and suprema that

$$a_\sigma\leqslant a_\tau\leqslant b_\tau\leqslant b_\sigma\quad \forall\tau\geqslant\sigma\geqslant 0\,; \tag{2.4}$$

in other terms, the nets $(a_\tau)_{\tau\geqslant 0}$ and $(b_\tau)_{\tau\geqslant 0}$ are monotone. We show in the next result that they actually converge.

LEMMA 2.5. *Suppose that* X *is a separable BMNSO space. Let* $\xi\colon \mathcal{T}_{\geqslant 0}\times\Omega\to X$ *be any precompact* θ*-stochastic process, and let* $(a_\tau)_{\tau\geqslant 0}$ *and* $(b_\tau)_{\tau\geqslant 0}$ *be, respectively, the lower and the upper tails of the pullback trajectories of* ξ*. Then* $(a_\tau)_{\tau\geqslant 0}$ *and* $(b_\tau)_{\tau\geqslant 0}$ *both converge* θ*-almost everywhere. Furthermore, setting*

$$a_\infty := \lim_{\tau\to\infty} a_\tau \quad \textit{and} \quad b_\infty := \lim_{\tau\to\infty} b_\tau\,,$$

we have

$$a_\sigma\leqslant a_\tau\leqslant a_\infty\leqslant b_\infty\leqslant b_\tau\leqslant b_\sigma\quad\forall\tau\geqslant\sigma\geqslant 0\,. \tag{2.5}$$

Proof. Fix $\omega\in\Omega$ arbitrarily such that $\beta_\xi^0(\omega)$ is precompact. We shall show that every sequence $(\tau_n)_{n\in\mathbb{N}}$ going to infinity in $\mathcal{T}_{\geqslant 0}$ has a subsequence along which $(a_\tau(\omega))_{\tau\geqslant 0}$ converges to the same $a_\infty(\omega)\in X$. Thus $(a_\tau(\omega))_{\tau\geqslant 0}$ must itself converge to $a_\infty(\omega)$.

Passing to a subsequence, if necessary, we may assume without loss of generality that $(\tau_n)_{n\in\mathbb{N}}$ is nondecreasing, so, $(a_{\tau_n}(\omega))_{n\in\mathbb{N}}$ is also nondecreasing in view of (2.4). Note that

$$a_{\tau_n}(\omega)\in\big(-\operatorname{shell}(-\beta_\xi^0(\omega))\big)\quad\forall n\in\mathbb{N}\,.$$

Since $\beta_\xi^0(\omega)$ is precompact by hypothesis, so is $-\beta_\xi^0(\omega)$, and it then follows from Theorem A.6 that $-\operatorname{shell}(-\beta_\xi^0(\omega))$ is compact. Hence $(a_{\tau_n}(\omega))_{n\in\mathbb{N}}$ converges to some $a_\infty(\omega) \in X$ by monotonicity [29, Lemma 1.2, p. 3]. Given any other sequence $(\sigma_n)_{n\in\mathbb{N}}$ going to infinity in $\mathcal{T}_{\geqslant 0}$, we may use the same argument, passing into a subsequence, if necessary, to conclude that $(a_{\sigma_n})_{n\in\mathbb{N}}$ converges to some $\widetilde{a}_\infty(\omega) \in X$. It remains to show that $a_\infty(\omega) = \widetilde{a}_\infty(\omega)$.

Choose subsequences $(k_n)_{n\in\mathbb{N}}$ and $(l_n)_{n\in\mathbb{N}}$ of $(n)_{n\in\mathbb{N}}$ such that

$$\tau_n \leqslant \sigma_{k_n} \quad \text{and} \quad \sigma_n \leqslant \tau_{l_n}\,,$$

and so

$$a_{\tau_n}(\omega) \leqslant a_{\sigma_{k_n}}(\omega) \quad \text{and} \quad a_{\sigma_n}(\omega) \leqslant a_{\tau_{l_n}}(\omega) \quad \forall n \in \mathbb{N}\,.$$

Taking the limit as n goes to infinity, we obtain

$$a_\infty(\omega) \leqslant \widetilde{a}_\infty(\omega) \quad \text{and} \quad \widetilde{a}_\infty(\omega) \leqslant a_\infty(\omega)\,,$$

showing that indeed $a_\infty(\omega) = \widetilde{a}_\infty(\omega)$.

Since $\beta_\xi^0(\omega)$ is precompact for θ-almost all $\omega \in \Omega$, a map $a_\infty\colon \Omega \to X$ is thus well-defined θ-almost everywhere by

$$a_\infty(\omega) := \lim_{\tau\to\infty} a_\tau(\omega)\,, \quad \omega \in \Omega\,.$$

In particular,

$$a_\infty := \lim_{n\to\infty} a_{\tau_n}$$

for any sequence $(\tau_n)_{n\in\mathbb{N}}$ going to infinity in $\mathcal{T}_{\geqslant 0}$. So, measurability follows from [19, Chapter 11, section 1, Property M7, p. 248].

The proof that $(b_\tau)_{\tau\geqslant 0}$ converges θ-almost everywhere to a random variable $b_\infty\colon \Omega \to X$ proceeds along the same lines. We obtain (2.5) by fixing $\tau \geqslant \sigma$ arbitrarily and taking the limit as τ goes to infinity in (2.4). ◻

Remark 2.6. The key step in the proof of the proposition above was the observation that $\operatorname{shell}(\beta_\xi^0(\omega))$ is compact. A simpler proof is possible in finite-dimensional spaces. The cone $V_+ \subseteq V$ is said to be *regular* if every monotone, order-bounded sequence converges in norm; that is, $(v_n)_{n\in\mathbb{N}}$ is convergent whenever

$$v_1 \leqslant v_2 \leqslant v_3 \leqslant \cdots \leqslant v_n \leqslant u$$

for some $u \in V$. It is not difficult to show that a cone in a finite-dimensional BMNSO space is always regular—in fact, only normality is needed. Since

$$a_{\tau_1}(\omega) \leqslant a_{\tau_2}(\omega) \leqslant a_{\tau_3}(\omega) \leqslant \cdots \leqslant a_{\tau_n}(\omega) \cdots \leqslant b_0(\omega)\,,$$

one could have then concluded the convergence of the sequence $(a_{\tau_n}(\omega))_{n\in\mathbb{N}}$ in a finite-dimensional space by appealing to regularity.

Lemma 2.5 above motivates the following definition of θ-lim sup and θ-lim inf in separable BMNSO spaces.

Definition 2.7 (θ-lim inf and θ-lim sup). *Given a separable BMNSO space X and a precompact θ-stochastic process $\xi\colon \mathcal{T}_{\geqslant 0} \times \Omega \to X$, we define $\theta\text{-}\underline{\lim}\,\xi$ to be the random variable $\Omega \to X$ defined for θ-almost all $\omega \in \Omega$ by*

$$[\theta\text{-}\underline{\lim}\,\xi](\omega) := \lim_{\tau\to\infty} \inf \beta_\xi^\tau(\omega)\,.$$

Similarly, we define $\theta\text{-}\overline{\lim}\,\xi$ to be the random variable $\Omega \to X$ defined for θ-almost all $\omega \in \Omega$ by

$$[\theta\text{-}\overline{\lim}\,\xi](\omega) := \lim_{\tau\to\infty} \sup \beta_\xi^\tau(\omega)\,.$$

Conversely, when we write $\theta\text{-}\underline{\lim}\,\xi$ or $\theta\text{-}\overline{\lim}\,\xi$ for some θ-stochastic process $\xi\colon \mathcal{T}_{\geqslant 0} \times \Omega \to X$, it will be clear from the context that ξ is precompact and that the symbols represent the random variables defined above.

It follows straight from the definition above that

$$\theta\text{-}\underline{\lim}\,\xi \leqslant \theta\text{-}\overline{\lim}\,\xi$$

for any precompact θ-stochastic process $\xi\colon \mathcal{T} \times \Omega \to X$. Moreover, we will have equality if and only if ξ converges pointwise.

Lemma 2.8. *Suppose that $\xi\colon \mathcal{T}_{\geqslant 0} \times \Omega \to X$ is a precompact θ-stochastic process on a separable BMNSO space X. Then*

$$\check{\xi}_t(\omega) \longrightarrow \xi_\infty(\omega) \quad \text{as} \quad t \to \infty \quad \widetilde{\forall}\omega \in \Omega \tag{2.6}$$

for some $\xi_\infty \in X_{\mathcal{B}}^\Omega$ if and only if

$$\theta\text{-}\underline{\lim}\,\xi = \theta\text{-}\overline{\lim}\,\xi = \xi_\infty =: \theta\text{-}\lim \xi\,. \tag{2.7}$$

Proof. ($\Leftarrow$) Suppose that (2.7) holds for some $\xi_\infty \in X_{\mathcal{B}}^\Omega$. Let $(a_\tau)_{\tau\geqslant 0}$ and $(b_\tau)_{\tau\geqslant 0}$ be a lower and an upper tail of the pullback trajectories of ξ, respectively. By definition, we have

$$a_\tau(\omega) \leqslant \check{\xi}_\tau(\omega) \leqslant b_\tau(\omega) \quad \widetilde{\forall}\omega \in \Omega\,, \quad \forall \tau \geqslant 0\,.$$

By (2.5), we have

$$a_\tau(\omega) \leqslant \xi_\infty(\omega) \leqslant b_\tau(\omega) \quad \widetilde{\forall}\omega \in \Omega\,, \quad \forall \tau \geqslant 0\,.$$

Thus by the triangle inequality and normality,

$$\begin{aligned} \|\check{\xi}_\tau(\omega) - \xi_\infty(\omega)\| &\leqslant \|\check{\xi}_\tau(\omega) - a_\tau(\omega)\| + \|\xi_\infty(\omega) - a_\tau(\omega)\| \\ &\leqslant 2C_{X_+}\|b_\tau(\omega) - a_\tau(\omega)\| \quad \widetilde{\forall}\omega \in \Omega\,, \quad \forall \tau \geqslant 0\,, \end{aligned}$$

where $C_{X_+} \geqslant 0$ is the normality constant of the underlying cone $X_+ \subseteq X$. By the hypothesis that $\theta\text{-}\underline{\lim}\,\xi = \theta\text{-}\overline{\lim}\,\xi$ and Lemma 2.5, it follows that $b_\tau - a_\tau \longrightarrow 0$ as $\tau \to \infty$ for θ-almost every $\omega \in \Omega$. Combining this with the inequality above, we obtain (2.6).

($\Rightarrow$) Now suppose (2.6) holds. Fix arbitrarily $\omega \in \Omega$ such that

$$\check{\xi}_t(\omega) \longrightarrow \xi_\infty(\omega) \quad \text{as} \quad t \to \infty\,. \tag{2.8}$$

Then it follows from Lemma A.2 that

$$a_\tau(\omega) = \inf_{t\geqslant\tau} \check{\xi}_t(\omega) \longrightarrow \xi_\infty(\omega) \quad \text{and} \quad b_\tau(\omega) = \sup_{t\geqslant\tau} \check{\xi}_t(\omega) \longrightarrow \xi_\infty(\omega)$$

as $\tau \to \infty$. Since (2.8) holds for θ-almost all $\omega \in \Omega$, we conclude that (2.7) also holds. $\square$

Naturally, inequalities are also preserved by θ-lim inf and θ-lim sup.

LEMMA 2.9. *Suppose that $\xi_1, \xi_2 \colon \mathcal{T}_{\geqslant 0} \times \Omega \to X$ are precompact θ-stochastic processes on a separable BMNSO space X. If $\xi_1 \leqslant \xi_2$, then*

$$\theta\text{-}\underline{\lim}\, \xi_1 \leqslant \theta\text{-}\underline{\lim}\, \xi_2 \quad \textit{and} \quad \theta\text{-}\overline{\lim}\, \xi_1 \leqslant \theta\text{-}\overline{\lim}\, \xi_2 \,.$$

Proof. We will carry out the details for $\theta\text{-}\overline{\lim}\, \xi_1 \leqslant \theta\text{-}\overline{\lim}\, \xi_2$. Let $(b_\tau^{(1)})_{\tau \geqslant 0}$ and $(b_\tau^{(2)})_{\tau \geqslant 0}$ be upper tails of the pullback trajectories of ξ_1 and ξ_2, respectively. Since

$$(\xi_1)_t(\omega) \leqslant (\xi_2)_t(\omega) \quad \widetilde{\forall} \omega \in \Omega \,, \quad \forall t \geqslant 0 \,,$$

it follows from θ-invariance that

$$(\xi_1)_t(\theta_{-t}\omega) \leqslant (\xi_2)_t(\theta_{-t}\omega) \quad \widetilde{\forall} \omega \in \Omega \,, \quad \forall t \geqslant 0 \,.$$

Hence, for each $\tau \geqslant 0$,

$$b_\tau^{(1)}(\omega) = \sup\{(\xi_1)_t(\theta_{-t}\omega)\,;\; t \geqslant \tau\} \leqslant \sup\{(\xi_2)_t(\theta_{-t}\omega)\,;\; t \geqslant \tau\} = b_\tau^{(2)}(\omega) \quad \widetilde{\forall} \omega \in \Omega \,.$$

By taking the limits as $\tau \to \infty$ in the inequality above, we obtain

$$\theta\text{-}\overline{\lim}\, \xi_1 = \lim_{\tau \to \infty} (\xi_1)_\tau \leqslant \lim_{\tau \to \infty} (\xi_2)_\tau = \theta\text{-}\overline{\lim}\, \xi_2 \,.$$

The other inequality can be proved using the same argument. ∎

2.3. Tempered convergence and continuity. As illustrated by various examples discussed throughout [6] and [23], ω-wise convergence in the pullback sense alone can sometimes be difficult to work with. It is often desirable to have some control over the growth of trajectories along the orbits of the group action θ. We now conceptualize a notion of *tempered convergence* and the notion of continuity derived from it.

A nonnegative, real-valued random variable $r \colon \Omega \to \mathbb{R}_{\geqslant 0}$ is said to be *tempered* if

$$K_{\gamma,\omega} := \sup_{s \in \mathcal{T}} r(\theta_s \omega)\, \mathrm{e}^{-\gamma |s|} < \infty \quad \forall \gamma > 0 \,, \quad \widetilde{\forall} \omega \in \Omega \,. \tag{2.9}$$

More generally, a random variable $R \colon \Omega \to X$ is said to be *tempered* if $r := \|R\| \colon \Omega \to \mathbb{R}_{\geqslant 0}$ is tempered in the sense above. We denote the family of tempered random variables $\Omega \to X$ by X_θ^Ω.

We note that the θ-invariant subset of full measure on which (2.9) holds can be constructed so as to be independent of $\gamma > 0$. We also note that the family X_θ^Ω of tempered random variables $\Omega \to X$ is a module over the family $\mathbb{R}_\theta^\Omega$ of real-valued tempered random variables, with operations of addition and scalar multiplication defined ω-wise. Finally, observe that if $r_1, r_2 \colon \Omega \to X$ are random variables such that $\|r_1\| \leqslant \|r_2\|$ and r_2 is tempered, then r_1 is also tempered.

DEFINITION 2.10 (tempered convergence). *We say that a net $(\xi_\alpha)_{\alpha \in A}$ in $X_\mathcal{B}^\Omega$* converges in the tempered sense *to a random variable $\xi_\infty \in X_\mathcal{B}^\Omega$ if there exists a nonnegative, tempered random variable $r \colon \Omega \to \mathbb{R}_{\geqslant 0}$ and an $\alpha_0 \in A$ such that*

(1) *$\xi_\alpha(\omega) \longrightarrow \xi_\infty(\omega)$ for θ-almost all $\omega \in \Omega$, and*

(2) *$\|\xi_\alpha(\omega) - \xi_\infty(\omega)\| \leqslant r(\omega)$ for all $\alpha \geqslant \alpha_0$, for θ-almost all $\omega \in \Omega$.*

In this case we write $\xi_\alpha \to_\theta \xi_\infty$.

DEFINITION 2.11 (tempered continuity). *A map*

$$\mathcal{K} \colon \mathcal{U} \subseteq U_\mathcal{B}^\Omega \longrightarrow X_\mathcal{B}^\Omega$$

is said to be tempered continuous *if $\mathcal{K}(u_\alpha) \to_\theta \mathcal{K}(u_\infty)$ for every net $(u_\alpha)_{\alpha \in A}$ in $\mathcal{U}$ such that $u_\alpha \to_\theta u_\infty$ for some $u_\infty \in \mathcal{U}$.*

2.4. Tempered trajectories.

Definition 2.12 (tempered trajectories). *A θ-stoschastic process $\xi \in \mathcal{S}_\theta^X$ is said to be* tempered *if there exists a nonnegative, tempered random variable $r\colon \Omega \to \mathbb{R}_{\geqslant 0}$ such that*

$$\|x\| \leqslant r(\omega) \quad \widetilde{\forall}\omega \in \Omega\,, \quad \forall x \in \beta_\xi^0(\omega)\,; \tag{2.10}$$

in other words,

$$\|\check{\xi}_t(\omega)\| = \|\xi_t(\theta_{-t}\omega)\| \leqslant r(\omega) \quad \widetilde{\forall}\omega \in \Omega\,, \quad \forall t \geqslant 0\,. \tag{2.11}$$

The family of all tempered θ-stochastic processes $\mathcal{T}_{\geqslant 0} \times \Omega \to X$ shall be denoted by $\mathcal{V}_\theta^X$.

Observe that, by virtue of θ-invariance, condition (2.11) is equivalent to

$$\|\xi_t(\omega)\| \leqslant r(\theta_t\omega) \quad \widetilde{\forall}\omega \in \Omega\,, \quad \forall t \geqslant 0\,.$$

The idea here is to have a term to talk about θ-stochastic processes which, as far as their oscillatory behavior is concerned, look somewhat like a θ-stationary process generated by a tempered random variable. Indeed, it is not difficult to show that the θ-stationary θ-stochastic process generated by a tempered variable is also tempered. Furthermore, any shift $\rho_s(\xi)$ of a tempered trajectory ξ is again a tempered trajectory, and any θ-concatenation $\xi\Diamond_s\zeta$ of tempered trajectories ξ and ζ is also tempered.

If ξ is a tempered trajectory, then it follows from (2.11) that $\beta_\xi^0(\omega)$ is bounded for θ-almost every $\omega \in \Omega$. Consequently, tempered trajectories are automatically precompact if the underlying space X is finite-dimensional. Note, however, that the converse of this statement is not necessarily true—a precompact trajectory need not be tempered, even in finite-dimensional spaces.

Proposition 2.13. *If $\xi \in \mathcal{S}_\theta^X$ is a tempered trajectory and $\xi_\infty\colon \Omega \to X$ is a map such that*

$$\xi_t(\theta_{-t}\omega) \longrightarrow \xi_\infty(\omega) \quad \text{as} \quad t \to \infty \quad \widetilde{\forall}\omega \in \Omega\,,$$

then ξ_∞ is a tempered random variable. Furthermore, in that case convergence is tempered.

Proof. It follows again from [19, Chapter 11, section 1, Property M7, p. 248] that ξ_∞ is measurable. (View ξ_∞ as the limit along a sequence $(t_n)_{n\in\mathbb{N}}$ going to infinity in $\mathcal{T}_{\geqslant 0}$.) Let $r\colon \Omega \to \mathbb{R}_{\geqslant 0}$ be a nonnegative, tempered random variable such that (2.11) holds. Then, by continuity of the norm,

$$\|\xi_\infty(\omega)\| = \lim_{t\to\infty} \|\xi_t(\theta_{-t}\omega)\| \leqslant r(\omega) \quad \widetilde{\forall}\omega \in \Omega\,.$$

Thus ξ_∞ is tempered. By the triangle inequality,

$$\|\xi_t(\theta_{-t}\omega) - \xi_\infty(\omega)\| \leqslant 2r(\omega) \quad \widetilde{\forall}\omega \in \Omega\,, \quad \forall t \geqslant 0\,.$$

Thus convergence occurs indeed in the tempered sense. ∎

Corollary 2.14. *Suppose that $\xi\colon \mathcal{T}_{\geqslant 0} \times \Omega \to X$ is a precompact, tempered θ-stochastic process on X. Then θ-$\underline{\lim}\,\xi$ and θ-$\overline{\lim}\,\xi$ are tempered random variables, and convergence of the tails $(a_\tau)_{\tau\geqslant 0}$ and $(b_\tau)_{\tau\geqslant 0}$ to θ-$\underline{\lim}\,\xi$ and θ-$\overline{\lim}\,\xi$, respectively, occur in the tempered sense.*

3. RDSIO. We are now ready to define the concept of RDSIO.

3.1. RDSI. We first discuss inputs.

DEFINITION 3.1 (θ-inputs). *We say that a subset* $\mathcal{U} \subseteq \mathcal{S}_\theta^U$ *is a* class of θ-inputs *if it has the following closure properties:*

(J1) $\rho_s(u) \in \mathcal{U}$ *for any* $u \in \mathcal{U}$ *and any* $s \geqslant 0$*, and*

(J2) $u \Diamond_s v \in \mathcal{U}$ *for any* $u, v \in \mathcal{U}$ *and any* $s \geqslant 0$.

Example 3.2 (θ-inputs). We discuss here several natural classes of θ-inputs.

(A) The family $\mathcal{V}_\theta^U$ of tempered θ-stochastic processes $\mathcal{T}_{\geqslant 0} \times \Omega \to U$ is a class of θ-inputs. Moreover, $U_\theta^\Omega \subseteq \mathcal{V}_\theta^U$, where we will identify, here and later, U_θ^Ω with the subset of $\mathcal{S}_\theta^U$ consisting of the θ-stationary θ-stochastic processes $\mathcal{T}_{\geqslant 0} \times \Omega \to U$ generated by tempered random variables $\Omega \to U$.

(B) The family $\mathcal{K}_\theta^U$ of precompact θ-stochastic processes $\mathcal{T}_{\geqslant 0} \times \Omega \to U$ also satisfies (J1) and (J2), thus constituting a class of θ-inputs as well. However, it is not necessarily true, in general, that $U_\theta^\Omega \subseteq \mathcal{K}_\theta^U$—even in finite dimensions.

(C) We introduce a third notable class of θ-inputs, namely, the family $\mathcal{S}_\infty^U$ consisting of all θ-stochastic processes $u \in \mathcal{S}_\theta^U$ such that

$$t \longmapsto |u_t(\omega)|\,, \quad t \geqslant 0\,,$$

is locally essentially bounded for each $\omega \in \Omega$. Note that

$$t \longmapsto u(\theta_t \omega)\,, \quad t \geqslant 0\,,$$

is locally essentially bounded for θ-almost every $\omega \in \Omega$ whenever u is a tempered random variable. Therefore $U_\theta^\Omega \subseteq \mathcal{S}_\infty^U$.

(D) Finally, note that the arbitrary intersection of classes of θ-inputs is a class of θ-inputs. In particular, $\mathcal{V}_\theta^U \cap \mathcal{K}_\theta^U$, $\mathcal{V}_\theta^U \cap \mathcal{S}_\infty^U$, $\mathcal{K}_\theta^U \cap \mathcal{V}_\theta^U$, and $\mathcal{V}_\theta^U \cap \mathcal{K}_\theta^U \cap \mathcal{S}_\infty^U$ are classes of θ-inputs. Furthermore, $U_\theta^\Omega \subseteq \mathcal{V}_\theta^U \cap \mathcal{S}_\infty^U$.

DEFINITION 3.3 (RDSI). *An* RDSI *is an ordered triple* $(\theta, \varphi, \mathcal{U})$ *consisting of an MPDS* $\theta = (\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathcal{T}})$*, a class of* θ*-inputs* $\mathcal{U} \subseteq \mathcal{S}_\theta^U$*, and a map*

$$\varphi : \mathcal{T}_{\geqslant 0} \times \Omega \times X \times \mathcal{U} \to X$$

satisfying

(I1) $\varphi_u := \varphi(\cdot, \cdot, \cdot, u) \colon \mathcal{T}_{\geqslant 0} \times \Omega \times X \to X$ *is* $(\mathcal{B}(\mathcal{T}_{\geqslant 0}) \otimes \mathcal{F} \otimes \mathcal{B})$*-measurable for each fixed* $u \in \mathcal{U}$*;*

(I2) $\varphi(t, \omega, \cdot, u) : X \to X$ *is continuous for each fixed* $(t, \omega, u) \in \mathcal{T}_{\geqslant 0} \times \Omega \times \mathcal{U}$*;*

(I3) $\varphi(0, \omega, x, u) = x$ *for each* $(\omega, x, u) \in \Omega \times X \times \mathcal{U}$*;*

(I4) *for any* $s, t \geqslant 0$*,* $\omega \in \Omega$*,* $x \in X$*, and* $u, v \in \mathcal{U}$*,*

$$[\varphi(s, \omega, x, u) = y \;\;\&\;\; \varphi(t, \theta_s \omega, y, v) = z] \quad \Rightarrow \quad z = \varphi(s + t, \omega, x, u \Diamond_s v)\,;$$

(I5) *given any* $t \geqslant 0$*,* $\omega \in \Omega$*,* $x \in X$*, and* $u, v \in \mathcal{U}$*, if* $u_\tau(\omega) = v_\tau(\omega)$ *for Lebesgue-almost all* $\tau \in [0, t)$*, then* $\varphi(t, \omega, x, u) = \varphi(t, \omega, x, v)$.

Remark 3.4. An immediate consequence of the above definition is that,

(I4′) for each arbitrarily fixed $s, t \geqslant 0$, $x \in X$, and $\omega \in \Omega$,

$$\varphi(t + s, \omega, x, u) \;=\; \varphi(t, \theta_s \omega, \varphi(s, \omega, x, u), \rho_s(u)) \quad \forall u \in \mathcal{U}\,.$$

Indeed, we have

$$u \Diamond_s \rho_s(u) = u \quad \forall u \in \mathcal{S}_\theta^U\,.$$

Thus (I4′) follows straight from (I4) with $v = \rho_s(u)$.

Example 3.5 (RDSI generated by RDEI).. In this example we shall give sufficient conditions for an RDEI,

$$\dot{\xi} = f(\theta_t\omega, \xi, u_t(\omega))\,, \quad t \geqslant 0\,, \quad \omega \in \Omega\,, \quad u \in \mathcal{U}\,,$$

to generate an RDSI. This effectively amounts to solving a family of ordinary differential equations (ODE) parametrized by $\omega \in \Omega$, with some special attention devoted to emerging measurability concerns, and can be done with ideas from standard existence and uniqueness theorems for ODE combined with textbook measure theory tools (for which we omit the details).

Given a Borel subset $U \subseteq \mathbb{R}^k$ and a set $\mathcal{U}$ of θ-inputs $\mathbb{R}_{\geqslant 0} \times \Omega \to U$, we shall say that an $(\mathcal{F} \otimes \mathcal{B}(\mathbb{R}^n) \otimes \mathcal{B}(U))$-measurable map $f\colon \Omega \times \mathbb{R}^n \times U \to \mathbb{R}^n$ is a *θ-righthand side* (*with respect to* $\mathcal{U}$) if

(R1) $f(\omega, \cdot, u)\colon \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz for every $\omega \in \Omega$ and every $u \in U$, and

(R2) for each $\omega \in \Omega$, every $u \in \mathcal{U}$, and any $b > a \geqslant 0$,

$$\int_a^b \|f(\theta_t\omega, \cdot, u_t(\omega))\|_K \, dt < \infty$$

for every compact $K \subseteq \mathbb{R}^n$.

Now suppose that $f\colon \Omega \times \mathbb{R}^n \times U \to \mathbb{R}^n$ is a θ-right-hand side with respect to $\mathcal{S}_\infty^U$, and suppose that f satisfies the growth condition

$$|f(\omega, x, u)| \leqslant \alpha(\omega)|x| + \beta(\omega) + c(u) \quad \forall \omega \in \Omega\,, \quad \forall (x, u) \in \mathbb{R}^n \times U\,, \tag{3.1}$$

for some tempered random variables $\alpha, \beta\colon \Omega \to \mathbb{R}_{\geqslant 0}$ and some continuous function $c\colon U \to \mathbb{R}_{\geqslant 0}$. Then the RDEI

$$\dot{\xi} = f(\theta_t\omega, \xi, u_t(\omega))\,, \quad t \geqslant 0\,, \quad \omega \in \Omega\,, \quad u \in \mathcal{S}_\infty^U\,, \tag{3.2}$$

generates an RDSI $(\theta, \varphi, \mathcal{S}_\infty^U)$ uniquely determined by the properties that

$$\varphi(0, \omega, x, u) = x \quad \forall (\omega, x, u) \in \Omega \times \mathbb{R}^n \times \mathcal{S}_\infty^U\,,$$

and

$$\frac{d}{dt}\varphi(t, \omega, x, u) = f(\theta_t\omega, \varphi(t, \omega, x, u), u_t(\omega)) \tag{3.3}$$

for each $(\omega, x, u) \in \Omega \times \mathbb{R}^n \times \mathcal{S}_\infty^U$, for Lebesgue-almost every $t \geqslant 0$.

A discrete time analogue of RDEI is given by a *random difference equation with inputs (RdEI)*

$$\xi^+ = \xi_{n+1} := f(\theta_n\omega, \xi_n, u_n(\omega))\,, \quad n \geqslant 0\,, \quad u \in \mathcal{S}_\theta^U\,,$$

which (uniquely) generates a (discrete) RDSI $(\theta, \varphi, \mathcal{S}_\theta^U)$, characterized by the property that

$$\varphi(n+1, \omega, x, u) \equiv f(\theta_n\omega, \varphi(n, \omega, x, u), u_n(\omega))\,.$$

The construction of the flow from the one-step transitions is a simple exercise; see details in [23].

An important subclass of RDEI (and RdEI) are RDSI generated by *linear* RDEI (and RdEI). We analyze the former in more detail in Example 3.15.

The concept of RDSI subsumes that of an RDS. Indeed, given an RDSI $(\theta, \varphi, \mathcal{U})$, fix arbitrarily a θ-stationary θ-input $\bar{u} \in \mathcal{U}$. Consistently with the convention that an overbar is used to indicate the θ-stationary process generated by a given random variable, we remove the bar to denote the random variable generating a given θ-stationary process—in other words, we denote by u the (θ-almost everywhere uniquely defined) random variable generating $\bar{u}$. We then define

$$\varphi_u := \varphi(\cdot, \cdot, \cdot, \bar{u})\colon \mathcal{T}_{\geqslant 0} \times \Omega \times X \longrightarrow X\,.$$

Then φ_u is a crude cocycle which can be perfected (see [23, Lemma 2.3 and Proposition 2.3, pp. 64–65] for details). Thus given an RDSI $(\theta, \varphi, \mathcal{U})$ and a θ-stationary input $u \in \mathcal{U}$, we shall always assume that φ_u has already been replaced by an indistinguishable perfection and then refer to the resulting RDS (θ, φ_u).

As in [23], we define the θ-stochastic processes $\xi^{x,u}\colon \mathcal{T}_{\geqslant 0} \times \Omega \to X$ by

$$\xi_t^{x,u}(\omega) := \varphi(t, \omega, x(\omega), u), \quad (t, \omega) \in \mathcal{T}_{\geqslant 0} \times \Omega \quad \forall x \in X_{\mathcal{B}}^{\Omega}\,, \quad \forall u \in \mathcal{U}\,.$$

Recall our convention that a check mark ˇ indicates the pullback of the θ-stochastic process being accented. Thus $\check{\xi}_t^{x,u}(\omega) \equiv \varphi(t, \theta_{-t}\omega, x(\theta_{-t}\omega), u)$.

DEFINITION 3.6 (I/S characteristic). *An* RDSI $(\theta, \varphi, \mathcal{U})$ *is said to have an* I/S characteristic $\mathcal{K}\colon U_\theta^\Omega \to X_\theta^\Omega$ if $U_\theta^\Omega \subseteq \mathcal{U}$ and

$$\check{\xi}_t^{x,u} \longrightarrow_\theta \mathcal{K}(u) \quad \text{as} \quad t \to \infty$$

for every $x \in X_\theta^\Omega$, for every $u \in U_\theta^\Omega$.

We next illustrate the definitions of RDSI and I/S characteristic with a few discrete time examples.

Iterated function systems in the sense of [4, Definition 1, p. 82] can be interpreted as RDS or RDSI (see also [3]). We illustrate how their limit fractals can be realized as the state characteristic associated with given θ-stationary θ-input.

Let $\theta = (\Omega, \mathcal{F}, \mathbb{P}, (\theta_n)_{n\in\mathcal{T}})$ be the (ergodic) MPDS defined as the "Bernoulli shift" on the probability space $(\Omega_0, \mathcal{F}_0, \mathbb{P}_0)$, where $\Omega_0 := \{1, 2, 3, 4\}$, $\mathcal{F}_0 := 2^{\Omega_0}$, and $\mathbb{P}_0\colon \mathcal{F}_0 \to [0, 1]$ is defined by $\mathbb{P}_0(\{1\}) := 0.01$, $\mathbb{P}_0(\{2\}) := 0.85$, $\mathbb{P}_0(\{3\}) := 0.07$, and $\mathbb{P}_0(\{4\}) := 0.07$; that is, $\Omega = \Omega_0^{\mathbb{Z}}$, $\mathcal{F}$ is the σ-algebra generated by the cylinder subsets of Ω, $\mathbb{P}$ is the canonical probability measure on $\mathcal{F}$, and $\theta_n\colon \Omega \to \Omega$ is the "shift n steps to the left" operator for each $n \in \mathbb{Z}$. Let $X := \mathbb{R}^2$ and $U := [0, 1] \times [0, 1]$, and consider the discrete RDSI $(\theta, \varphi, \mathcal{S}_\theta^U)$ generated by the RdEI

$$\xi^+ = A(\theta_n\omega)\xi + u_n(\omega)\,, \quad n \geqslant 0\,, \quad u \in \mathcal{S}_\theta^U\,, \tag{3.4}$$

where $A\colon \Omega \to M_{2\times 2}(\mathbb{R})$ is defined as follows. First define $A_0\colon \Omega_0 \to M_{2\times 2}(\mathbb{R})$ by

$$A_0(1) := \begin{bmatrix} 0 & 0 \\ 0 & 0.16 \end{bmatrix}, \quad A_0(2) := \begin{bmatrix} 0.85 & 0.04 \\ -0.04 & 0.85 \end{bmatrix},$$

$$A_0(3) := \begin{bmatrix} 0.2 & -0.26 \\ 0.23 & 0.22 \end{bmatrix}, \quad \text{and} \quad A_0(4) := \begin{bmatrix} -0.15 & 0.28 \\ 0.26 & 0.24 \end{bmatrix},$$

then define A by setting $A(\omega) := A_0(\omega_0)$, $\omega \in \Omega$. The largest singular values of $A_0(1)$, $A_0(2)$, $A_0(3)$, and $A_0(4)$ can be numerically estimated to be, respectively, $\sigma_0^{max}(1) = 0.16$, $\sigma_0^{max}(2) \approx 0.8509$, $\sigma_0^{max}(3) \approx 0.3407$, and $\sigma_0^{max}(4) \approx 0.3792$. Since

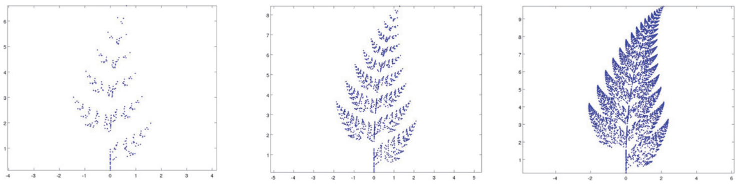

FIG. 3. *Tempered, pullback convergence of φ to $\mathcal{K}(u)$, starting at $x = 0$, at times $t = 6, 10, 18$.*

$A(\omega) \equiv A_0(\omega_0)$, the largest singular value of $A(\omega)$ is $\sigma^{max}(\omega) \equiv \sigma_0^{\max}(\omega_0)$ for each $\omega \in \Omega$, and therefore

$$\left\| \prod_{j=s}^{s+r-1} A(\theta_j \omega) \right\| \leqslant (\sigma_0^{max}(2))^r \quad \forall \omega \in \Omega\,, \quad \forall s \in \mathbb{Z}\,, \quad \forall r \geqslant 0\,, \tag{3.5}$$

where $\|\cdot\|$ is the operator norm on a square matrix with respect to the Euclidean norm on vectors (see [31, section A.2, pp. 448–451]). With this estimate, one can show that φ has a continuous I/S characteristic $\mathcal{K}\colon U_\theta^\Omega \to X_\theta^\Omega$ given by

$$[\mathcal{K}(u)](\omega) = \sum_{j=-\infty}^{-1} \left(\prod_{k=j+1}^{-1} A(\theta_k \omega) \right) u(\theta_j \omega) \quad \forall u \in U_\theta^\Omega\,, \quad \widetilde{\forall} \omega \in \Omega\,.$$

This follows along the same lines as the the continuous-time analogue (see Example 3.15 below and [23, Example 2.3, pp. 71–76]). We briefly remark that, in order for the I/S characteristic to exist in a linear example such as this one, it is sufficient that $\mathbb{E}\sigma^{max} < 1$. It then follows from the multiplicative ergodic theorem, once again as in the continuous-time analogue, that (3.5) holds with the right-hand side replaced by $\gamma(\theta_s \omega)\lambda^r$ for some $\lambda \in (0,1)$ and some nonnegative, tempered random variable $\gamma\colon \Omega \to \mathbb{R}_{\geqslant 0}$ (see Remark 3.16 below).

Now consider the θ-stationary input $u \in U_\theta^\Omega$ defined as follows. First define

$$u_0(1) := \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \;\; u_0(2) := \begin{bmatrix} 0 \\ 1.6 \end{bmatrix}, \;\; u_0(3) := \begin{bmatrix} 0 \\ 1.6 \end{bmatrix}, \;\; \text{and} \quad u_0(4) := \begin{bmatrix} 0 \\ 0.44 \end{bmatrix},$$

then set $u(\omega) := u_0(\omega_0)$, $\omega \in \Omega$. The support of the image of $\mathcal{K}(u)$ is the *Barnsley fern* [4, Table 3.8.3, p. 87, and Figure 3.8.3, p. 92]. Figure 3 shows the results after steps $n = 6$, $n = 10$, and $n = 18$ of a simulation of the pullback trajectories of φ starting at $x = 0$ and subject to the input u defined above.

Next, consider a variation of this example. Let θ be an MPDS defined as above, except having instead $\mathbb{P}_0(\{1\}) := 0.10$, $\mathbb{P}_0(\{2\}) := 0.35$, $\mathbb{P}_0(\{3\}) := 0.35$, and $\mathbb{P}_0(\{4\}) := 0.20$. Define $A\colon \Omega \to M_{\mathbb{R}\times\mathbb{R}}(2)$ as above via

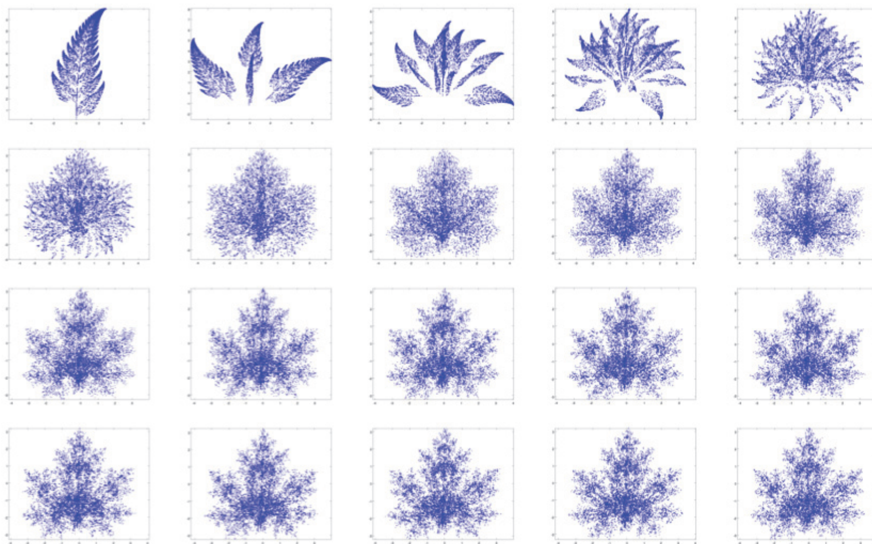

FIG. 4. *Barnsley fern into maple leaf (iteration is top left to bottom right).*

$$A_0(1) := \begin{bmatrix} 0.14 & 0.01 \\ 0 & 0.51 \end{bmatrix}, \quad A_0(2) := \begin{bmatrix} 0.43 & 0.52 \\ -0.45 & 0.5 \end{bmatrix},$$
$$A_0(3) := \begin{bmatrix} 0.45 & -0.49 \\ 0.47 & -1.62 \end{bmatrix}, \quad \text{and} \quad A_0(4) := \begin{bmatrix} 0.49 & 0 \\ 0 & 0.51 \end{bmatrix},$$

and consider the (discrete) RDSI $(\theta, \varphi, \mathcal{S}_\theta^U)$ generated by (3.4). It can be shown as before, by looking at the largest singular values of A, that this RDSI has a continuous I/S characteristic. The state characteristic corresponding to the θ-stationary input $u \in U_\theta^\Omega$ defined as above via

$$u_0(1) := \begin{bmatrix} -0.08 \\ -1.31 \end{bmatrix}, \ u_0(2) := \begin{bmatrix} 1.49 \\ -0.75 \end{bmatrix}, \ u_0(3) := \begin{bmatrix} -1.62 \\ -0.74 \end{bmatrix}, \ \text{and} \quad u_0(4) := \begin{bmatrix} 0.02 \\ 1.62 \end{bmatrix}$$

is a distribution over the *maple leaf*. Figure 4 illustrates the (pullback) convergence to this state characteristic starting from the distribution over the Barnsley fern given in the previous example. In this way, one translates into the formalism of RDSI a "controllability" problem between pictures: using the input in question, we have steered the fern into the maple leaf.

The next two definitions are growth conditions on the (pullback) long-term behavior of RDSI.

DEFINITION 3.7 (tempered RDSI). *An* RDSI $(\theta, \varphi, \mathcal{U})$ *is said to be* tempered *if the θ-stochastic processes $\xi^{x,u}$ are tempered for every tempered initial state $x \in X_\theta^\Omega$ and every tempered input $u \in \mathcal{U}$.*

In the context of the above definition, given any tempered initial state $x \in X_\theta^\Omega$ and any tempered input $u \in \mathcal{U}$, we have

$$\|\check{\xi}_t^{x,u}(\omega)\| = \|\varphi(t, \theta_{-t}\omega, x(\theta_{-t}\omega), u)\| \leqslant r(\omega) \quad \forall t \geqslant 0\,, \quad \widetilde{\forall}\omega \in \Omega\,,$$

for some nonnegative, tempered random variable r. Thus $\check{\xi}_t^{x,u}$ is a tempered random variable for each $t \geqslant 0$.

Definition 3.8 (compact RDSI). *An* RDSI $(\theta, \varphi, \mathcal{U})$ *is said to be* compact *if the θ-stochastic processes $\xi^{x,u}$ are precompact for every tempered initial state $x \in X_\theta^\Omega$ and every precompact input $u \in \mathcal{U}$.*

Remark 3.9. Although the context here is somewhat different, this definition is related to the concept of *compact* RDS given in [6, Definition 1.4.3, p. 30]. Chueshov does not require the "entering time" $t_0(\omega)$ to be uniform in ω, while we do require the entering time in Definition 2.2 to be $t = 0$ for θ-almost every $\omega \in \Omega$. On the other hand, Chueshov requires the "absorbing set" to be the same for every initial state, while we allow for it to depend on $x \in X_\theta^\Omega$.

3.2. Monotone RDSI. Suppose that $(X, \leqslant)$ is a partially ordered space. For any $a, b \in X_\mathcal{B}^\Omega$, we write $a \leqslant b$ to mean that $a(\omega) \leqslant b(\omega)$ for θ-almost all $\omega \in \Omega$. Similarly, for any $p, q \in \mathcal{S}_\theta^X$, we write $p \leqslant q$ to mean that $p_t(\omega) \leqslant q_t(\omega)$ for all $t \geqslant 0$, for θ-almost all $\omega \in \Omega$. Taking into account the identification of θ-almost everywhere equal maps discussed in subsection 2.1, this convention induces partial orders in $X_\mathcal{B}^\Omega$ and $\mathcal{S}_\theta^X$.

Definition 3.10 (monotone RDSI). *An* RDSI $(\theta, \varphi, \mathcal{U})$ *is said to be* monotone *if the underlying state and input spaces are partially ordered spaces $(X, \leqslant_X)$ and $(U, \leqslant_U)$, and*

$$\varphi(\cdot, \cdot, x(\cdot), u) \leqslant_X \varphi(\cdot, \cdot, z(\cdot), v)$$

whenever $x, z \in X_\mathcal{B}^\Omega$ and $u, v \in \mathcal{U}$ are such that $x \leqslant_X z$ and $u \leqslant_U v$.

Definition 3.11 (Monotone Characteristics). *Suppose $(X, \leqslant_X)$ and $(U, \leqslant_U)$ are partially ordered spaces. A map $\mathcal{M}\colon E \subseteq U_\mathcal{B}^\Omega \to X_\mathcal{B}^\Omega$ is said to be* monotone *or* order-preserving *if $\mathcal{M}(u) \leqslant_X \mathcal{M}(v)$ whenever $u, v \in E$ satisfy $u \leqslant_U v$. Analogously, if $\mathcal{M}(u) \geqslant_X \mathcal{M}(v)$ whenever $u \leqslant_U v$, then $\mathcal{M}$ is said to be* antimonotone *or* order-reversing*.*

Most of the time, the underlying partially ordered space will be clear from the context. So, unless there is any risk of confusion, we shall again drop the indices in "$\leqslant_X$" and "$\leqslant_U$" and write simply "$\leqslant$."

As in the deterministic scenario, if an RDSI $(\theta, \varphi, \mathcal{U})$ is monotone and has an I/S characteristic $\mathcal{K}\colon U_\theta^\Omega \to X_\theta^\Omega$, then $\mathcal{K}$ is order-preserving; in other words, if $u, v \in U_\theta^\Omega$ and $u \leqslant v$, then $\mathcal{K}(u) \leqslant \mathcal{K}(v)$.

Let U be a Borel subset of $\mathbb{R}^k$ and $f\colon \Omega \times \mathbb{R}^n \times U \to \mathbb{R}^n$ be a θ-right-hand side, with respect to $\mathcal{S}_\infty^U$, satisfying growth condition (3.1). Then the RDEI

$$\dot{\xi} = f(\theta_t\omega, \xi, u_t(\omega)), \quad t \geqslant 0, \quad \omega \in \Omega, \quad u \in \mathcal{S}_\infty^U, \tag{3.6}$$

generates an RDSI $(\theta, \varphi, \mathcal{S}_\infty^U)$, as we noted in Example 3.5 above. The proposition below, which is essentially a ω-wise version of its deterministic analogue, gives sufficient conditions for this RDSI to be monotone.

Proposition 3.12 (Kamke conditions). *Suppose that $\mathbb{R}^n$ and $\mathbb{R}^k$ are partially ordered by their respective positive orthant cones, that $U \subseteq \mathbb{R}^k$ is closed, order-convex and has nonempty interior, and that $f\colon \Omega \times \mathbb{R}^n \times U \to \mathbb{R}^n$ is a θ-right-hand side such that $f(\omega, \cdot, \cdot)\colon \mathbb{R}^n \times U \to \mathbb{R}^n$ is continuously differentiable for θ-almost all $\omega \in \Omega$. Then the RDSI $(\theta, \varphi, \mathcal{S}_\infty^U)$ generated by the RDEI* (3.6) *is monotone if and only if, for θ-almost every ω,*

(K1) $\frac{\partial f_i}{\partial x_j}(x, u) \geqslant 0$ *for every $x \in \mathbb{R}^n$, every $u \in \operatorname{int} U$, and every $i, j \in \{1, \ldots, n\}$ such that $i \neq j$, and*

(K2) $\frac{\partial f_i}{\partial u_j}(x,u) \geqslant 0$ *for every* $x \in \mathbb{R}^n$*, every* $u \in \operatorname{int} U$*, every* $i \in \{1,\ldots,n\}$*, and every* $j \in \{1,\ldots,k\}$.

Proof. This follows from [1, Proposition III.2, p. 1687], applied for each $\omega \in \Omega$ such that the conditions hold. ∎

3.3. The CICS property. Suppose an RDSI $(\theta,\varphi,\mathcal{U})$ has a continuous I/S characteristic $\mathcal{K}$, and then suppose that the pullback of a tempered θ-input $u \in \mathcal{S}^U_\infty$ converges (in the tempered sense) to a $u_\infty \in U^\Omega_\theta$. Since

$$\check{\xi}^{x,u_\infty}(\omega) \longrightarrow_\theta [\mathcal{K}(u_\infty)](\omega)\,, \quad \text{as} \quad t \to \infty\,, \quad \forall x \in X^\Omega_\theta\,,$$

one may expect that the continuity of φ (with respect to the state variable) and $\mathcal{K}$ would imply that

$$\check{\xi}^{x,u}_t(\omega) \longrightarrow_\theta [\mathcal{K}(u_\infty)](\omega)\,, \quad \text{as} \quad t \to \infty\,, \quad \forall x \in X^\Omega_\theta\,. \tag{3.7}$$

Unfortunately this is not true in general. In fact, this CICS property might fail even in the deterministic case, as illustrated by the counterexample in [28]. Other hypotheses such as asymptotic stability of the state characteristics or monotonicity of the flow are needed.

The CICS result below (Corollary 3.14) was first stated and proved for deterministic and finite-dimensional "monotone control systems" by Angeli and Sontag [1, Proposition V.5(2)]. In [12, Theorem 1], Enciso and Sontag explored normality to extend the result to infinite-dimensional systems. Replacing the geometric properties in [12] by minihedrality, it is possible to extend this result further to monotone RDSI.

We shall derive Corollary 3.14 from a more general result, given in Theorem 3.13 below. Note that if the input u converges in the pullback sense, then it is precompact. But if we know a priori that u is precompact, then the θ-limits θ-$\underline{\lim}\, u$ and θ-$\overline{\lim}\, u$ exist, even if u does not converge (in the pullback sense). If the θ-limits θ-$\underline{\lim}\, \xi^{x,u}$ and θ-$\overline{\lim}\, \xi^{x,u}$ also exist, then it is natural to ask how they may compare with $\mathcal{K}(\theta\text{-}\underline{\lim}\, u)$ and $\mathcal{K}(\theta\text{-}\overline{\lim}\, u)$.

THEOREM 3.13 (sub-CICS). *Suppose that X and U are separable BMNSO spaces. Let $(\theta,\varphi,\mathcal{U})$ be a tempered, compact, monotone RDSI with state space X and input space U and suppose that φ has a continuous I/S characteristic $\mathcal{K}\colon U^\Omega_\theta \to X^\Omega_\theta$. Then*

$$\mathcal{K}(\theta\text{-}\underline{\lim}\, u) \leqslant \theta\text{-}\underline{\lim}\, \xi^{x,u} \quad \textit{and} \quad \theta\text{-}\overline{\lim}\, \xi^{x,u} \leqslant \mathcal{K}(\theta\text{-}\overline{\lim}\, u)$$

for every $x \in X^\Omega_\theta$ and every tempered, precompact $u \in \mathcal{U}$.

Proof. We work out the details for the first inequality, the proof of the second one being entirely analogous. Fix arbitrarily a tempered initial state $x \in X^\Omega_\theta$ and a tempered, precompact input $u \in \mathcal{U}$. By Definitions 3.7 and 3.8, the θ-stochastic process $\xi^{x,u}$ is also tempered and precompact. Let $(a_\tau)_{\tau \geqslant 0}$ be the lower tail of the pullback trajectories of u. From Corollary 2.14, both θ-$\underline{\lim}\, u$ and θ-$\underline{\lim}\, \xi^{x,u}$ exist and define tempered random variables in their respective spaces. Furthermore,

$$a_\tau \longrightarrow_\theta \theta\text{-}\underline{\lim}\, u \quad \text{as} \quad \tau \to \infty\,.$$

Therefore $\mathcal{K}(a_\tau) \longrightarrow_\theta \mathcal{K}(\theta\text{-}\underline{\lim}\, u)$ as $\tau \to \infty$ by continuity. So, it is enough to show that

$$\mathcal{K}(a_{\tau_n}) \leqslant \theta\text{-}\underline{\lim}\, \xi^{x,u} \quad \forall n \in \mathbb{N} \tag{3.8}$$

for an arbitrarily fixed sequence $(\tau_n)_{n\in\mathbb{N}}$ going to infinity in $[0,\infty)$.

Fix arbitrarily $n \in \mathbb{N}$. Let $\bar{a}_{\tau_n}$ be the θ-stationary process generated by a_{τ_n}. We claim that

(D1) $\bar{a}_{\tau_n} \leqslant \rho_{\tau_n}(u)$, and
(D2) $\check{\xi}_t^{x,u} = \check{\xi}_{t-\tau_n}^{\check{\xi}_{\tau_n}^{x,u},\rho_{\tau_n}(u)}$ for all $t \geqslant \tau_n$.

Assuming (D1) and (D2), we have

$$\xi_s^{\check{\xi}_{\tau_n}^{x,u},\bar{a}_{\tau_n}} \leqslant \xi_s^{\check{\xi}_{\tau_n}^{x,u},\rho_{\tau_n}(u)} \quad \forall s \geqslant 0$$

by monotonicity. Now $\bar{a}_{\tau_n}$ is a tempered θ-stochastic process, since a_{τ_n} is a tempered random variable. Likewise, $\rho_{\tau_n}(u)$ is precompact and tempered. Finally, $\check{\xi}_{\tau_n}^{x,u}$ is also tempered, since it is the "pullback slice" of a tempered trajectory. We conclude that $\xi_s^{\check{\xi}_{\tau_n}^{x,u},\bar{a}_{\tau_n}}$ is tempered and that $\xi_s^{\check{\xi}_{\tau_n}^{x,u},\rho_{\tau_n}(u)}$ is precompact and tempered. It follows from the existence of the I/S characteristic and Lemmas 2.8 and 2.9 that

$$\mathcal{K}(a_{\tau_n}) = \theta\text{-}\underline{\lim}\, \xi^{\check{\xi}_{\tau_n}^{x,u},\bar{a}_{\tau_n}} \leqslant \theta\text{-}\underline{\lim}\, \xi^{\check{\xi}_{\tau_n}^{x,u},\rho_{\tau_n}(u)} = \theta\text{-}\underline{\lim}\, \xi^{x,u}\,.$$

Since $n \in \mathbb{N}$ was chosen arbitrarily, this proves (3.8).

It remains to prove (D1) and (D2). They each follow straight from the relevant definitions. Indeed, for any $t \geqslant 0$ and any $\omega \in \Omega$ for which $\bar{a}_{\tau_n}(\omega)$ is defined, we have

$$[\bar{a}_{\tau_n}]_t(\omega) = a_{\tau_n}(\theta_t\omega) = \inf_{s \geqslant \tau_n} u_s(\theta_{-s}\theta_t\omega) \leqslant u_{\tau_n+t}(\theta_{-(\tau_n+t)}\theta_t\omega) = [\rho_{\tau_n}(u)]_t(\omega)\,.$$

This proves (D1). For any $t \geqslant \tau_n$ and any $\omega \in \Omega$, we have

$$\begin{aligned}
\check{\xi}_t^{x,u}(\omega) &= \varphi(t-\tau_n+\tau_n, \theta_{-(t-\tau_n)-\tau_n}\omega, x(\theta_{-(t-\tau_n)-\tau_n}\omega), u) \\
&= \varphi(t-\tau_n, \theta_{-(t-\tau_n)}\omega, \varphi(\tau_n, \theta_{-\tau_n}\theta_{-(t-\tau_n)}\omega, x(\theta_{-\tau_n}\theta_{-(t-\tau_n)}\omega), u), \rho_{\tau_n}(u)) \\
&= \varphi(t-\tau_n, \theta_{-(t-\tau_n)}\omega, \check{\xi}_{\tau_n}^{x,u}(\theta_{-(t-\tau_n)}\omega), \rho_{\tau_n}(u))
\end{aligned}$$

by (I4′). This establishes (D2). □

Corollary 3.14 (random CICS). *Assume the same hypotheses as in Theorem 3.13. If $u \in \mathcal{U}$ is tempered and*

$$\check{u}_t \longrightarrow_\theta u_\infty \quad as \quad t \to \infty \tag{3.9}$$

for some $u_\infty \in U_\theta^\Omega$, then

$$\check{\xi}_t^{x,u} \longrightarrow_\theta \mathcal{K}(u_\infty) \quad as \quad t \to \infty \quad \forall x \in X_\theta^\Omega\,. \tag{3.10}$$

Proof. Fix arbitrarily $x \in X_\theta^\Omega$. As noted above, the tempered convergence in (3.9) implies u is precompact. So, by Theorem 3.13 and Lemma 2.8,

$$\mathcal{K}(u_\infty) \leqslant \theta\text{-}\underline{\lim}\, \xi^{x,u} \leqslant \theta\text{-}\overline{\lim}\, \xi^{x,u} \leqslant \mathcal{K}(u_\infty)\,.$$

It follows again by Lemma 2.8 that

$$\theta\text{-}\underline{\lim}\, \xi^{x,u} = \theta\text{-}\overline{\lim}\, \xi^{x,u} = \theta\text{-}\lim \xi^{x,u} = \mathcal{K}(u_\infty)\,,$$

yielding (3.10). □

Example 3.15 (linear RDSI). Set $X := \mathbb{R}^n$, $U := \mathbb{R}^k$, and $\mathcal{U} := \mathcal{S}_\infty^U$. Suppose that

$$A\colon \Omega \longrightarrow M_{n\times n}(\mathbb{R}) \quad \text{and} \quad B\colon \Omega \longrightarrow M_{n\times k}(\mathbb{R})$$

are random matrices such that

$$t \longmapsto A(\theta_t\omega)\,, \quad t \geqslant 0\,, \quad \text{and} \quad t \longmapsto B(\theta_t\omega)\,, \quad t \geqslant 0\,,$$

are locally essentially bounded for each $\omega \in \Omega$. Then the RDEI

$$\dot{\xi} = A(\theta_t\omega)\xi + B(\theta_t\omega)u_t(\omega)\,, \quad t \geqslant 0\,, \quad \omega \in \Omega\,, \quad u \in \mathcal{U}\,,$$

generates an RDSI $(\theta, \varphi, \mathcal{U})$, since the hypotheses in Example 3.5 are satisfied. We will present sufficient conditions for this RDSI to be tempered and precompact and to possess a continuous I/S characteristic. Further down we will also discuss hypotheses guaranteeing that the RDSI is monotone (with respect to the positive orthant cone-induced partial order).

Assume that

(L1) B is tempered, and

(L2) there exist a $\lambda > 0$ and a nonnegative, tempered random variable $\gamma \in (\mathbb{R}_{\geqslant})^{\Omega}_{\theta}$ such that the fundamental matrix solution

$$\Xi\colon \mathbb{R} \times \mathbb{R} \times \Omega \longrightarrow M_{n\times n}(\mathbb{R})$$

of

$$\dot{\xi} = A(\theta_t\omega)\xi\,, \quad t \geqslant 0\,, \quad \omega \in \Omega\,,$$

satisfies

$$\|\Xi(s, s + r, \omega)\| \leqslant \gamma(\theta_s\omega)\,\mathrm{e}^{-\lambda r} \quad \widetilde{\forall}\omega \in \Omega\,, \quad \forall s \in \mathbb{R}\,, \quad \forall r \geqslant 0\,.$$

Then $(\theta, \varphi, \mathcal{U})$ is tempered and has a tempered-continuous I/S characteristic $\mathcal{K}\colon U^{\Omega}_{\theta} \to X^{\Omega}_{\theta}$, given by

$$[\mathcal{K}(u)](\omega) = \int_{-\infty}^{0} \Xi(\sigma, 0, \omega)B(\theta_\sigma\omega)u(\theta_\sigma\omega)\,d\sigma \quad \forall u \in U^{\Omega}_{\theta}\,, \quad \widetilde{\forall}\omega \in \Omega\,.$$

Since this system is evolving in a finite-dimensional space, precompactness follows from temperedness, as noted before. We refer to [23, Example 2.3, pp. 71–76] for the details.

We now discuss monotonicity. Equip $\mathbb{R}^n$ and $\mathbb{R}^k$ with their respective positive orthant cone-induced partial orders. Let $X := \mathbb{R}^n_{\geqslant 0}$ and $U := \mathbb{R}^k_{\geqslant 0}$, which are closed order-intervals. If all off-diagonal entries of $A(\omega)$ are nonnegative for θ-almost every $\omega \in \Omega$, and all entries of $B(\omega)$ are nonnegative for θ-almost every $\omega \in \Omega$, then it follows from Proposition 3.10 that $(\theta, \varphi, \mathcal{S}^{U}_{\infty})$ is monotone.

Remark 3.16. If $\|A(\cdot)\| \in L^1(\Omega, \mathcal{F}, \mathbb{P})$, the largest eigenvalue $\overline{\lambda}(\cdot)$ of the Hermitian part of $A(\cdot)$ is such that

$$\mathbb{E}\overline{\lambda} := \int_{\Omega} \overline{\lambda}(\omega)\,d\mathbb{P}(\omega) < 0\,,$$

and the underlying MPDS θ is ergodic, then it follows from [6, Theorem 2.1.2, p. 60] that (L2) holds with $\lambda := -(\mathbb{E}\overline{\lambda} + \epsilon)$ for any choice of $\epsilon \in (0, -\mathbb{E}\overline{\lambda})$.

3.4. Output functions. We want to consider the RDE

$$\dot{\xi} = A(\theta_t\omega)\xi + B(\theta_t\omega)h(\theta_t\omega, \xi)\,, \quad t \geqslant 0\,, \quad \omega \in \Omega\,, \tag{3.11}$$

for several classes of nonlinearity $h\colon \Omega \times \mathbb{R}^n_{\geqslant 0} \to \mathbb{R}^k_{\geqslant 0}$. In each of Examples 4.5–4.7 below, we will apply Theorem 4.4 to show that the RDS generated by (3.11) has a unique, globally attracting, positive equilibrium. Our approach is to realize (3.11) as the "closed loop" of the RDSI from Example 3.15 and its "output function" h. In this section we will make these ideas precise.

Definition 3.17 (output functions). *An* output function *is a $(\mathcal{F} \otimes \mathcal{B}(X))$-measurable map $h\colon \Omega \times X \to Y$ into a separable BMNSO space Y such that*

$$h(\omega, \cdot)\colon X \longrightarrow Y$$

is continuous for each $\omega \in \Omega$. In this context Y is called an output space*.*

Definition 3.18 (RDSIO). *An* RDSIO *is a quadruple $(\theta, \varphi, \mathcal{U}, h)$ such that $(\theta, \varphi, \mathcal{U})$ is an* RDSI *and h is an output function.*

The Ω-component in the domain of output functions is important. It allows for the concept to model uncertainties in the readout as well.

It can be shown that if $D\colon \Omega \to 2^X \backslash \{\varnothing\}$ is a random set in X, then

$$\omega \longmapsto h(\omega, D(\omega)) := \{h(\omega, x)\,;\ x \in D(\omega)\}\,, \quad \omega \in \Omega\,,$$

is a random set in Y. In particular, if D is compact, then so is $h(\cdot, D(\cdot))$. (Refer to Remark 2.4.)

Given an output function h, we define its *induced output characteristic* $h_*\colon X_{\mathcal{B}}^{\Omega} \to Y_{\mathcal{B}}^{\Omega}$ by

$$[h_*(x)](\omega) := h(\omega, x(\omega))\,, \quad \omega \in \Omega\,,$$

for each $x \in X_{\mathcal{B}}^{\Omega}$. This is the natural way to map random states $x \in X_{\mathcal{B}}^{\Omega}$ into random readouts $y \in Y_{\mathcal{B}}^{\Omega}$, generalizing what is accomplished by the output function $h\colon X \to Y$ itself in the deterministic setting.

In the context of "closed-loop systems," "cascades," and "feedback interconnections," we shall be particularly interested in output funcions h such that $h_*(X_\theta^\Omega) \subseteq Y_\theta^\Omega$.

Definition 3.19 (temperedness preserving outputs). *An output function*

$$h\colon \Omega \times X \longrightarrow Y$$

is said to preserve temperedness *if the random set*

$$h(\cdot, D(\cdot))\colon \Omega \longrightarrow 2^Y \backslash \{\varnothing\}$$

is tempered for every tempered random set $D\colon \Omega \to 2^X \backslash \{\varnothing\}$.

In particular,

$$\omega \longmapsto h(\omega, x(\omega)) \quad \omega \in \Omega\,,$$

defines a tempered random variable $\Omega \to Y$ whenever $x\colon \Omega \to X$ is also a tempered random variable. Therefore $h_*(X_\theta^\Omega) \subseteq Y_\theta^\Omega$ whenever h is a temperedness preserving output function.

In examples and applications, temperedness preservation will typically arise as a consequence of growth conditions on the output function. For instance, if

(G1) there exist $M_0, M_1 \in (\mathbb{R}_{\geqslant 0})_\theta^\Omega$ and $n \in \mathbb{N}$ such that

$$\|h(\omega, x)\| \leqslant M_0(\omega) + M_1(\omega)\|x\|^n \quad \forall x \in X, \quad \widetilde{\forall}\omega \in \Omega$$

(*tempered polynomial growth*),

then h is temperedness preserving.

If h is a temperedness preserving output function, then it is not difficult to show that the θ-stochastic process $\eta^\xi \colon \mathcal{T}_{\geqslant 0} \times \Omega \to Y$ defined by

$$\eta_t^\xi(\omega) := h(\theta_t\omega, \xi_t(\omega)), \quad (t,\omega) \in \mathcal{T}_{\geqslant 0} \times \Omega,$$

is tempered whenever $\xi \colon \mathcal{T}_{\geqslant 0} \times \Omega \to X$ is a tempered θ-stochastic process in X. Furthermore, the restriction $h_*\big|_{X_\theta^\Omega} \colon X_\theta^\Omega \to Y_\theta^\Omega$ of the induced output characteristic to X_θ^Ω is tempered continuous.

Definition 3.20 (I/O characteristic). *Suppose that an* RDSIO $(\theta, \varphi, \mathcal{U}, h)$ *is such that the underlying* RDSI $(\theta, \varphi, \mathcal{U})$ *has an* I/S characteristic $\mathcal{K} \colon U_\theta^\Omega \to X_\theta^\Omega$, and the output function h preserves temperedness. Then the induced output characteristic $h_* \colon X_\theta^\Omega \to Y_\theta^\Omega$ of $(\theta, \varphi, \mathcal{U}, h)$ is well-defined, and so the map

$$\mathcal{K}^Y := h_* \circ \mathcal{K} \colon U_\theta^\Omega \longrightarrow Y_\theta^\Omega$$

is also well-defined. In this case the system is said to have an *input-to-output (I/O) characteristic* and, accordingly, $\mathcal{K}^Y$ is referred to as the *I/O characteristic* of the system.

In the particular case when $Y = U$, the I/O characteristic is an operator on the space U_θ^Ω of tempered random variables $\Omega \to U$. This operator can be informally interpreted as the "gain" of the system, a measure of how much a θ-stationary "signal" u changes when "processed" by the system.

Definition 3.21 (monotone and antimonotone outputs). *Let* $(X, \leqslant_X)$ *and* $(Y, \leqslant_Y)$ *be partially ordered spaces. An output function* $h \colon \Omega \times X \to Y$ *is said to be* monotone *if*

$$\widetilde{\forall}\omega \in \Omega, \quad x_1 \leqslant_X x_2 \quad \Rightarrow \quad h(\omega, x_1) \leqslant_Y h(\omega, x_2).$$

Analogously, if

$$\widetilde{\forall}\omega \in \Omega, \quad x_1 \leqslant_X x_2 \quad \Rightarrow \quad h(\omega, x_1) \geqslant_Y h(\omega, x_2),$$

then h is said to be antimonotone*.*

As usual, the underlying partial order will be clear from the context and we shall use simply $\leqslant$ to denote either of $\leqslant_X$ or $\leqslant_Y$. Furthermore, whenever we refer to a "monotone RDSI," an "order-preserving map," etc., the underlying spaces will be tacitly understood to be partially ordered. Note that the induced output characteristic h_* is order-preserving (order-reversing) whenever $h \colon \Omega \times X \to Y$ is a monotone (antimonotone) output function.

Definition 3.22 (closed-loop trajectory). *A θ-stochastic process $\xi \in \mathcal{S}_\theta^X$ is said to be a* closed-loop trajectory *of an* RDSIO $(\theta, \varphi, \mathcal{U}, h)$ *(starting at ξ_0) if*

(1) $Y = U$,

(2) *the θ-stochastic process $\eta^\xi \colon \mathcal{T}_{\geqslant 0} \times \Omega \longrightarrow U$ defined by*

$$\eta_t^\xi(\omega) := h(\theta_t\omega, \xi_t(\omega)), \quad t \geqslant 0, \quad \omega \in \Omega,$$

belongs to $\mathcal{U}$, and

(3) $\xi_t(\omega) = \varphi(t, \omega, \xi_0(\omega), \eta^\xi)$ *for all $t \geqslant 0$ and all $\omega \in \Omega$.*

Property (1) is quite natural. It does not make sense to talk about feeding the output of the system back into it, thus "closing the loop," if the output and input spaces do not coincide. The θ-stochastic process η^ξ defined in property (2) is the "readout" of the (θ-stochastic) trajectory ξ on the state space. Naturally, we can feed this readout as an input to the system only if it is itself an admissible θ-input. Property (3) then states that the original trajectory ξ could be recovered by letting the system evolve starting at ξ_0 and subject to the θ-input η^ξ.

4. The small-gain theorem. Let $\xi\colon \mathcal{T}_{\geqslant 0} \times \Omega \to X$ be a precompact, tempered closed-loop trajectory of an RDSIO $(\theta, \varphi, \mathcal{U}, h)$ such that its underlying RDSI $(\theta, \varphi, \mathcal{U})$ satisfies the hypotheses of Corollary 3.14. Let $\eta^\xi\colon \mathcal{T}_{\geqslant 0} \times \Omega \to U$ be the corresponding (tempered) output trajectory along ξ; that is,

$$\eta_t^\xi(\omega) = h(\theta_t\omega, \xi_t(\omega)) \quad \forall (t, \omega) \in \mathcal{T}_{\geqslant 0} \times \Omega\,.$$

If $\check{\eta}^\xi \to_\theta u_\infty$ for some $u_\infty \in \mathcal{U}$, then it follows from Corollary 3.14 that $\xi \to_\theta \mathcal{K}(u_\infty)$. When will it be true that $\check{\eta}^\xi \to_\theta u_\infty$ for some $u_\infty \in \mathcal{U}$? Are there reasonable conditions under which $\check{\eta}^\xi \to_\theta u_\infty$ for the same u_∞, for any (precompact, tempered) closed-loop trajectories, thus yielding convergence of any closed-loop trajectory of the system to $\mathcal{K}(u_\infty)$?

4.1. Result. In order to address the questions above, we first look at η^ξ for monotone or antimonotone output functions. It turns out that monotonicity alone already imposes severe constraints on the behavior of η^ξ.

LEMMA 4.1. *Suppose that $(\theta, \varphi, \mathcal{U}, h)$ is a monotone RDSIO possessing a continuous I/S characteristic $\mathcal{K}\colon U_\theta^\Omega \to X_\theta^\Omega$ and a monotone or antimonotone temperedness preserving output function h. Given a precompact, tempered closed-loop trajectory $\xi \in \mathcal{K}_\theta^X \cap \mathcal{V}_\theta^X$ of $(\theta, \varphi, \mathcal{U}, h)$, let $\eta^\xi \in \mathcal{U}$ be the corresponding (precompact, tempered) output trajectory along ξ; that is,*

$$\eta_t^\xi(\omega) = h(\theta_t\omega, \xi_t(\omega)) \quad \forall (t, \omega) \in \mathcal{T}_{\geqslant 0} \times \Omega\,.$$

Let $(a_\tau)_{\tau \geqslant 0}$ and $(b_\tau)_{\tau \geqslant 0}$ be, respectively, the lower and upper tails of the pullback trajectories of η^ξ, and let $\mathcal{K}^Y\colon U_\theta^\Omega \to U_\theta^\Omega$ be the I/O characteristic of $(\theta, \varphi, \mathcal{U}, h)$. Then

$$(\mathcal{K}^Y)^{2k}(a_\tau) \leqslant \theta\text{-}\underline{\lim}\,\eta^\xi \leqslant \theta\text{-}\overline{\lim}\,\eta^\xi \leqslant (\mathcal{K}^Y)^{2k}(b_\tau) \quad \forall k \in \mathbb{N}\,, \quad \forall \tau \geqslant 0\,.$$

This observation, which we shall prove further down, motivates the *small-gain condition*. Suppose that there exists an $u_\infty \in U_\theta^\Omega$ such that

$$\lim_{k\to\infty} [(\mathcal{K}^Y)^{2k}(a_\tau)](\omega) = u_\infty(\omega) = \lim_{k\to\infty} [(\mathcal{K}^Y)^{2k}(b_\tau)](\omega) \quad \widetilde{\forall}\omega \in \Omega\,.$$

Then it follows from Lemma 4.1 (together with Lemma 2.8 and Corollary 2.14) that $\check{\eta} \to_\theta u_\infty$, thus leaving us in the setting of Corollary 3.14.

DEFINITION 4.2 (small-gain condition). *We say that an* RDSIO $(\theta, \varphi, \mathcal{U}, h)$ *satisfying the hypotheses of Lemma* 4.1 *satisfies the* small-gain condition *if there exists a (necessarily unique) $u_\infty \in U_\theta^\Omega$ such that*

$$[(\mathcal{K}^Y)^k(u)](\omega) \longrightarrow u_\infty(\omega)$$

as $k \to \infty$ for θ-almost all $\omega \in \Omega$, for each $u \in U_\theta^\Omega$.

Remark 4.3. Note that we do not ask that convergence in the small-gain condition be tempered.

A small-gain theorem for RDS now follows almost effortlessly.

THEOREM 4.4 (small-gain theorem). *Suppose that $(\theta, \varphi, \mathcal{U}, h)$ is a tempered, monotone RDSIO possessing a continuous I/S characteristic $\mathcal{K}\colon U_\theta^\Omega \to X_\theta^\Omega$ and a monotone or antimonotone temperedness preserving output function h. Suppose, in addition, that $(\theta, \varphi, \mathcal{U}, h)$ satisfies the small-gain condition, and let $u_\infty \in U_\theta^\Omega$ be as in Definition 4.2. Then*

$$\check{\xi} \longrightarrow_\theta \mathcal{K}(u_\infty)$$

for every precompact, tempered closed-loop trajectory $\xi \in \mathcal{K}_\theta^X \cap \mathcal{V}_\theta^X$ *of* $(\theta, \varphi, \mathcal{U}, h)$*; in other words, every precompact, tempered closed-loop trajectory of* $(\theta, \varphi, \mathcal{U}, h)$ *converges* (*in the pullback, tempered sense*) *to* $\mathcal{K}(u_\infty)$.

Proof. Consider the notation introduced in the statement of Lemma 4.1. From the lemma,

$$(\mathcal{K}^Y)^{2k}(a_\tau) \leqslant \theta\text{-}\underline{\lim}\,\eta^\xi \leqslant \theta\text{-}\overline{\lim}\,\eta^\xi \leqslant (\mathcal{K}^Y)^{2k}(b_\tau) \quad \forall k \in \mathbb{N}\,, \quad \forall \tau \geqslant 0\,.$$

By the small-gain condition,

$$\lim_{k\to\infty}[(\mathcal{K}^Y)^{2k}(a_\tau)](\omega) = u_\infty(\omega) = \lim_{k\to\infty}[(\mathcal{K}^Y)^{2k}(b_\tau)](\omega) \quad \widetilde{\forall}\omega \in \Omega\,.$$

We obtain $\theta\text{-}\underline{\lim}\,\eta^\xi = u_\infty = \theta\text{-}\overline{\lim}\,\eta^\xi$, thus yielding $\check{\eta}^\xi \longrightarrow_\theta u_\infty$ via Lemma 2.8 and Proposition 2.11. It then follows from Corollary 3.14 that

$$\check{\xi} = \check{\xi}^{\xi_0,\eta^\xi} \longrightarrow_\theta \mathcal{K}(u_\infty)\,,$$

completing the proof. ▯

4.2. Examples. We now consider a few examples illustrating how Theorem 4.4 may be applied to establish the existence of unique, globally attracting equilibria for some classes of nonmonotone, nonlinear RDS generated by RDE. One may allude to the example in the introduction, namely, a biochemical circuit as illustrated in Figure 1, as a prototype for the more general examples discussed in what follows. As outlined in the introduction, this biochemical circuit may be modeled by an RDE,

$$\dot{\xi}_i \;=\; a_i(\theta_t\omega)\xi_i + \frac{b_i(\theta_t\omega)}{\beta_i(\theta_t\omega) + g_i(\xi_{i-1})}\,, \quad i = 1, 2, 3$$

(with the convention $\xi_0 = \xi_3$), for some nondecreasing functions $g_i \colon \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$. This can be written in matrix notation as $\dot{\xi} = A(\theta_t\omega)\xi + B(\theta_t\omega)h(\theta_t\omega, \xi)$, where

$$A(\omega) \equiv \operatorname{diag}(a_1(\omega), a_2(\omega), a_3(\omega))\,, \quad B(\omega) \equiv \operatorname{diag}(b_1(\omega), b_2(\omega), b_3(\omega))$$

and

$$h(\omega, \xi) \equiv \begin{bmatrix} \dfrac{1}{\beta_1(\omega) + g_1(\xi_3)} & \dfrac{1}{\beta_2(\omega) + g_2(\xi_1)} & \dfrac{1}{\beta_3(\omega) + g_3(\xi_2)} \end{bmatrix}^T . \tag{4.1}$$

So, consider the RDS (θ, φ) generated by an RDE

$$\dot{\xi} = A(\theta_t\omega)\xi + B(\theta_t\omega)h(\theta_t\omega, \xi)\,, \quad t \geqslant 0\,, \quad \omega \in \Omega\,,$$

where A and B are as in Example 3.15, and $h \colon \Omega \times \mathbb{R}^n \to \mathbb{R}^k$ is an output function. As discussed in the example, the RDSI $(\theta, \varphi, \mathcal{U})$ generated by the RDEI

$$\dot{\xi} = A(\theta_t\omega)\xi + B(\theta_t\omega)u_t(\omega)\,, \quad t \geqslant 0\,, \quad \omega \in \Omega\,, \quad u \in \mathcal{S}_\infty^U\,, \tag{4.2}$$

is tempered, monotone, and has a continuous I/S characteristic. Thus the burden of satisfying the hypotheses of Theorem 4.4 has now fallen all on h—the RDS generated by (4.2) will have a unique, globally attracting equilibrium whenever h is a monotone or antimonotone temperedness preserving output function such that the RDSIO $(\theta, \varphi, \mathcal{U}, h)$ satisfies the small-gain condition.

Example 4.5 (saturated readouts). Consider an output function $h\colon \Omega \times \mathbb{R}^n_{\geqslant 0} \to \mathbb{R}^k_{\geqslant 0}$ defined by

$$h(\omega, x) := \left(\frac{\alpha_j(\omega)}{\beta_j(\omega) + g_j(x)} \right)_{j=1}^{k}, \quad (\omega, x) \in \Omega \times \mathbb{R}^n_{\geqslant 0},$$

where $\alpha, \beta\colon \Omega \to \mathbb{R}^k_{\geqslant 0}$ and $g\colon \mathbb{R}^n_{\geqslant 0} \to \mathbb{R}^k_{\geqslant 0}$ satisfy the following hypotheses:

(P1) α and β are continuous and uniformly bounded away from zero and infinity along θ-almost every orbit; more precisely, for θ-almost every $\omega \in \Omega$, $t \mapsto \alpha(\theta_t\omega) \in \mathbb{R}^k$, $t \in \mathbb{R}$, and $t \mapsto \beta(\theta_t\omega) \in \mathbb{R}^k$, $t \in \mathbb{R}$ are continuous, and there exist an $\epsilon = (\epsilon_1, \ldots, \epsilon_k = \epsilon(\omega) \gg 0$ and an $M = (M_1, \ldots, M_k) = M(\omega) \geqslant 0$ such that $\epsilon \leqslant \alpha(\theta_t\omega), \beta(\theta_t\omega) \leqslant M$ for all $t \in \mathbb{R}$;

(P2) g is continuous, order-preserving, sublinear (see section 5), and bounded.

Observe that (4.1) is a special case, when all components of α are equal to 1.

It follows straight from the monotonicity of g in (P2) that h is antimonotone. From (P1),

$$0 \leqslant h(\theta_s\omega, x(\theta_s\omega)) \leqslant \frac{M(\omega)}{\epsilon(\omega)} \quad \forall s \in \mathbb{R}, \quad \forall \omega \in \Omega,$$

for any $x \in X^\Omega_\theta$, where the quotient is defined coordinatewise. In particular, h preserves temperedness. It remains to check that the I/O characteristic $\mathcal{K}^Y$ of $(\theta, \varphi, \mathcal{S}^U_\infty, h)$ satisfies the small-gain condition.

For each $u \in U^\Omega_\theta$,

$$[\mathcal{K}^Y(u)](\omega) = \left(\frac{\alpha_j(\omega)}{\beta_j(\omega) + g_j\big([\mathcal{K}(u)](\omega)\big)} \right)_{j=1}^{k} \qquad \widetilde{\forall} \omega \in \Omega.$$

Fix arbitrarily such an u, and fix arbitrarily any $\omega \in \Omega$ for which

$$[\mathcal{K}(u)](\omega) = \int_{-\infty}^{0} \Xi(\sigma, 0, \omega) B(\theta_\sigma\omega) u(\theta_\sigma\omega)\, d\sigma$$

is well-defined and (P1) holds. For each $t \in \mathbb{R}$, we have

$$[\mathcal{K}^Y(u)](\theta_t\omega) = \left(\frac{\alpha_j(\theta_t\omega)}{\beta_j(\theta_t\omega) + g_j\left(\int_{-\infty}^{t} \Xi(\sigma, t, \omega) B(\theta_\sigma\omega) u(\theta_\sigma\omega)\, d\sigma \right)} \right)_{j=1}^{k}$$

by a simple, linear change of variables. Set

$$A_\omega := A(\theta.\omega), B_\omega := B(\theta.\omega), \alpha_\omega := \alpha(\theta.\omega), \text{and } \beta_\omega := \beta(\theta.\omega),$$

and consider the operator $\mathcal{H}_\omega\colon L^\theta_+(\mathbb{R}^k) \to L^\theta_+(\mathbb{R}^k)$ defined by

$$[\mathcal{H}_\omega(\nu)](t) := \left(\frac{(\alpha_\omega)_j(t)}{(\beta_\omega)_j(t) + g_j\left(\int_{-\infty}^{t} \Xi_\omega(\sigma, t) B_\omega(\sigma)\nu(\sigma)\, d\sigma \right)} \right)_{j=1}^{k}, \quad t \in \mathbb{R},$$

for each $\nu \in L_+^\theta(\mathbb{R}^k)$, where $L_+^\theta(\mathbb{R}^k)$ is the family of paths $\mu\colon \mathbb{R} \to \mathbb{R}^k_{\geqslant 0}$ such that

$$\forall \gamma > 0\,, \quad \sup_{s\in\mathbb{R}} |\mu(s)|\, \mathrm{e}^{-\gamma|s|} < \infty\,,$$

and $\Xi_\omega\colon \mathbb{R}\times\mathbb{R} \to M_{n\times n}(\mathbb{R})$ is the fundamental solution of the linear system of ODE $\dot{\xi} = A_\omega(t)\xi$, $t \in \mathbb{R}$. By equipping $L_+^\theta(\mathbb{R}^k)$ with the partial order naturally induced by the positive orthant cone-induced partial order in $\mathbb{R}^k$, we may introduce the "Thompson metric," with respect to which $\mathcal{H}_\omega$ can be shown (see section 5) to have a unique, globally attracting fixed point $u_\omega \in \operatorname{int} L_+^\infty(\mathbb{R}^k) \subseteq L_+^\theta(\mathbb{R}^k)$. In fact, the representative u_ω can be chosen to be continuous, and in this case we have pointwise convergence:

$$\lim_{m\to\infty} [\mathcal{H}_\omega^m(\nu)](t) = u_\omega(t) \quad \forall t \in \mathbb{R}\,, \quad \forall \nu \in L_+^\theta(\mathbb{R}^k)\,.$$

(See Proposition 5.9.)

We now show that the map $u_\infty\colon \Omega \to U$ defined by $u_\infty(\omega) := u_\omega(0)$, $\omega \in \Omega$, belongs to U_θ^Ω and is the unique, globally attracting fixed point of $\mathcal{K}^Y$. Fix arbitrarily $u \in U_\theta^\Omega$. Then

$$\lim_{m\to\infty} [(\mathcal{K}^Y)^m(u)](\omega) = \lim_{m\to\infty} [\mathcal{H}_\omega^m(u(\theta.\omega))](0) = u_\omega(0) = u_\infty(\omega) \quad \widetilde{\forall}\omega \in \Omega\,.$$

In particular, u_∞ is the θ-almost sure, pointwise limit of measurable maps

$$\omega \longmapsto [(\mathcal{K}^Y)^m(u)](\omega)\,, \quad \omega \in \Omega\,, \quad m = 1, 2, 3, \dots\,,$$

hence itself is measurable. Fix arbitrarily any $\omega \in \Omega$ for which the limit above holds. By the uniqueness of the continuous representatives u_ω, we have

$$u_\infty(\theta_t\omega) = u_{\theta_t\omega}(0) = u_\omega(t) \quad \forall t \in \mathbb{R}\,.$$

Therefore $t \mapsto u_\infty(\theta_t\omega)$, $t \in \mathbb{R}$, is bounded. In particular,

$$\sup_{t\in\mathbb{R}} |u_\infty(\theta_t\omega)|\, \mathrm{e}^{-\gamma|t|} < \infty \quad \forall \gamma > 0\,.$$

We conclude that u_∞ is tempered and a fixed point of $\mathcal{K}^Y$. Since $u \in U_\theta^\Omega$ was chosen arbitrarily, this also shows that u_∞ is globally attractive.

Example 4.6 (unbounded g). Now consider an output function $h\colon \Omega \times \mathbb{R}^n \to \mathbb{R}^k$ defined by

$$h(\omega, x) := \left(\frac{\alpha_j(\omega) + \widetilde{\alpha}_j(\omega) g_j(x)}{\beta_j(\omega) + \widetilde{\beta}_j(\omega) g_j(x)}\right)_{j=1}^k\,, \quad (\omega, x) \in \Omega \times \mathbb{R}^n\,,$$

where $\alpha, \widetilde{\alpha}, \beta, \widetilde{\beta}\colon \Omega \to \mathbb{R}^k_{>0}$, and $g\colon \mathbb{R}^n_{\geqslant 0} \to \mathbb{R}^k_{\geqslant 0}$ satisfy the following:

(P1′) α, $\widetilde{\alpha}$, β, and $\widetilde{\beta}$ are continuous and uniformly bounded away from zero along the orbit of ω, and satisfy $\alpha_j(\theta_t\omega)/\beta_j(\theta_t\omega) \geqslant \widetilde{\alpha}_j(\theta_t\omega)/\widetilde{\beta}_j(\theta_t\omega)$ for all $t \in \mathbb{R}$, $j = 1, \dots, k$, for θ-almost every $\omega \in \Omega$, and

(P2′) g is continuous, order-preserving, and sublinear.

Then h is antimonotone and temperedness preserving, and the I/O characteristic $\mathcal{K}^Y$ of $(\theta, \varphi, \mathcal{S}_\infty^U, h)$ satisfies the small-gain condition. This follows along the same lines of Example 4.5 by invoking Proposition 5.10.

Example 4.7 (periodic θ). In Example 4.5, suppose that the underlying MPDS θ is T-periodic; that is, there exists $T > 0$ such that $\theta_{t+T}\omega = \theta_t\omega$ for all $t \in \mathbb{R}$ $\widetilde{\forall}\omega \in \Omega$.

Then g need not be bounded in order for the small-gain condition to be satisfied. This also follows along the lines of Example 4.5.

Naturally each of the continuous-time examples above has a discrete-time counterpart. We omit the details, which can be found in [21].

4.3. Proof of Lemma 4.1. The remainder of this section is concerned with the proof of Lemma 4.1.

LEMMA 4.8. *Assume the same hypotheses as in* Lemma 4.1.

(1) *If h is monotone, then* $h_*(\theta\text{-}\underline{\lim}\,\xi) \leqslant \theta\text{-}\underline{\lim}\,\eta^\xi \leqslant \theta\text{-}\overline{\lim}\,\eta^\xi \leqslant h_*(\theta\text{-}\overline{\lim}\,\xi)$.

(2) *If h is antimonotone, then* $h_*(\theta\text{-}\overline{\lim}\,\xi) \leqslant \theta\text{-}\underline{\lim}\,\eta^\xi \leqslant \theta\text{-}\overline{\lim}\,\eta^\xi \leqslant h_*(\theta\text{-}\underline{\lim}\,\xi)$.

Proof. Since $\theta\text{-}\underline{\lim}\,\eta^\xi \leqslant \theta\text{-}\overline{\lim}\,\eta^\xi$ holds automatically (see the observation following Definition 2.7), we need only prove the four outer inequalities. The argument for each of them goes along the same lines, so we shall provide the details for only one of the inequalities. Namely, we assume that h is antimonotone and prove that

$$h_*(\theta\text{-}\overline{\lim}\,\xi) \leqslant \theta\text{-}\underline{\lim}\,\eta^\xi\,.$$

Let $(\beta_\tau)_{\tau\geqslant 0}$ be the upper tail of the pullback trajectories of ξ. Since

$$\xi_t(\theta_{-t}\omega) \leqslant \beta_\tau(\omega) \quad \widetilde{\forall}\omega \in \Omega\,, \quad \forall t \geqslant \tau \geqslant 0\,,$$

it follows from the antimonotonicity of h that

$$h(\omega, \xi_t(\theta_{-t}\omega)) \geqslant h(\omega, \beta_\tau(\omega)) \quad \widetilde{\forall}\omega \in \Omega\,, \quad \forall t \geqslant \tau \geqslant 0\,.$$

Let $(a_\tau)_{\tau\geqslant 0}$ be the lower tail of the pullback trajectories of η^ξ. For every $\tau \geqslant 0$, we have

$$a_\tau(\omega) = \inf_{t\geqslant\tau} \eta^\xi_t(\theta_{-t}\omega) = \inf_{t\geqslant\tau} h(\omega, \xi_t(\theta_{-t}\omega)) \geqslant h(\omega, \beta_\tau(\omega)) = [h_*(\beta_\tau)](\omega) \quad \widetilde{\forall}\omega \in \Omega\,.$$

Since h_* is tempered continuous, by letting $\tau \to \infty$ in the chain of equalities and inequalities above, we obtain

$$\theta\text{-}\underline{\lim}\,\eta^\xi = \lim_{\tau\to\infty} a_\tau \geqslant \lim_{\tau\to\infty} h_*(\beta_\tau) = h_*(\theta\text{-}\overline{\lim}\,\xi)\,.$$

As noted above, the proofs of the other inequalities are entirely analogous. ∎

Proof of Lemma 4.1. Since $(\theta, \varphi, \mathcal{U})$ is monotone and

$$a_\tau \leqslant \theta\text{-}\underline{\lim}\,\eta^\xi \leqslant \theta\text{-}\overline{\lim}\,\eta^\xi \leqslant b_\tau \quad \forall \tau \geqslant 0\,,$$

the I/S characteristic $\mathcal{K}$ is also monotone, and so

$$\mathcal{K}(a_\tau) \leqslant \mathcal{K}(\theta\text{-}\underline{\lim}\,\eta^\xi) \leqslant \mathcal{K}(\theta\text{-}\overline{\lim}\,\eta^\xi) \leqslant \mathcal{K}(b_\tau) \quad \forall \tau \geqslant 0\,.$$

By Theorem 3.13,

$$\mathcal{K}(\theta\text{-}\underline{\lim}\,\eta^\xi) \leqslant \theta\text{-}\underline{\lim}\,\xi \leqslant \theta\text{-}\overline{\lim}\,\xi \leqslant \mathcal{K}(\theta\text{-}\overline{\lim}\,\eta^\xi)\,. \tag{4.3}$$

Combining these with the previous inequalities, we obtain

$$\mathcal{K}(a_\tau) \leqslant \theta\text{-}\underline{\lim}\,\xi \leqslant \theta\text{-}\overline{\lim}\,\xi \leqslant \mathcal{K}(b_\tau) \quad \forall \tau \geqslant 0\,. \tag{4.4}$$

Suppose first that h is monotone. Then the induced output characteristic h_* preserves the inequalities in (4.4):

$$\mathcal{K}^Y(a_\tau) \leqslant h_*(\theta\text{-}\underline{\lim}\,\xi) \leqslant h_*(\theta\text{-}\overline{\lim}\,\xi) \leqslant \mathcal{K}^Y(b_\tau) \quad \forall \tau \geqslant 0\,.$$

By Lemma 4.8(1) below, we now have

$$\mathcal{K}^Y(a_\tau) \leqslant \theta\text{-}\underline{\lim}\,\eta^\xi \leqslant \theta\text{-}\overline{\lim}\,\eta^\xi \leqslant \mathcal{K}^Y(b_\tau) \quad \forall \tau \geqslant 0\,.$$

Now suppose that we have shown that

$$(\mathcal{K}^Y)^k(a_\tau) \leqslant \theta\text{-}\underline{\lim}\,\eta^\xi \leqslant \theta\text{-}\overline{\lim}\,\eta^\xi \leqslant (\mathcal{K}^Y)^k(b_\tau) \quad \forall \tau \geqslant 0 \tag{4.5}$$

for some $k \in \mathbb{N}$. Then, again, combining the monotonicity of $\mathcal{K}$ and h_*, (4.3), and Lemma 4.8(1), we obtain

$$\mathcal{K}\big((\mathcal{K}^Y)^k(a_\tau)\big) \leqslant \mathcal{K}(\theta\text{-}\underline{\lim}\,\eta^\xi) \leqslant \theta\text{-}\underline{\lim}\,\xi \leqslant \theta\text{-}\overline{\lim}\,\xi \leqslant \mathcal{K}(\theta\text{-}\overline{\lim}\,\eta^\xi) \leqslant \mathcal{K}\big((\mathcal{K}^Y)^k(b_\tau)\big)\,,$$

hence

$$(\mathcal{K}^Y)^{k+1}(a_\tau) \leqslant h_*(\theta\text{-}\underline{\lim}\,\xi) \leqslant \theta\text{-}\underline{\lim}\,\eta^\xi \leqslant \theta\text{-}\overline{\lim}\,\eta^\xi \leqslant h_*(\theta\text{-}\overline{\lim}\,\xi) \leqslant (\mathcal{K}^Y)^{k+1}(b_\tau)$$

for every $\tau \geqslant 0$. It follows by induction that (4.5) holds for every $k \in \mathbb{N}$. In particular, the conclusion of the lemma holds.

If h is antimonotone, then h_* is order-reversing. Thus applying h_* to each term in the chain of inequalities in (4.4) yields

$$\mathcal{K}^Y(b_\tau) \leqslant h_*(\theta\text{-}\overline{\lim}\,\xi) \leqslant h_*(\theta\text{-}\underline{\lim}\,\xi) \leqslant \mathcal{K}^Y(a_\tau) \quad \forall \tau \geqslant 0\,.$$

Applying Lemma 4.8(2), we get

$$\mathcal{K}^Y(b_\tau) \leqslant \theta\text{-}\underline{\lim}\,\eta^\xi \leqslant \theta\text{-}\overline{\lim}\,\eta^\xi \leqslant \mathcal{K}^Y(a_\tau) \quad \forall \tau \geqslant 0\,. \tag{4.6}$$

Applying $\mathcal{K}$ to each term in (4.6) and using (4.3) once again, we get

$$\mathcal{K}\big(\mathcal{K}^Y(b_\tau)\big) \leqslant \theta\text{-}\underline{\lim}\,\xi \leqslant \theta\text{-}\overline{\lim}\,\xi \leqslant \mathcal{K}\big(\mathcal{K}^Y(a_\tau)\big) \quad \forall \tau \geqslant 0\,.$$

Applying h_* to each term in the inequalities above and using Lemma 4.8(2) once again to simplify, we then get

$$(\mathcal{K}^Y)^2(a_\tau) \leqslant \theta\text{-}\underline{\lim}\,\eta^\xi \leqslant \theta\text{-}\overline{\lim}\,\eta^\xi \leqslant (\mathcal{K}^Y)^2(b_\tau) \quad \forall \tau \geqslant 0\,.$$

The argument can now be completed by induction on k just as in the previous case, using the monotonicity of $\mathcal{K}$, the antimonotonicity of h_*, (4.3), and Lemma 4.8(2) to simplify the two terms in the middle after each application of $\mathcal{K}$ and h_*, respectively.

5. Discrete iterations and the Thompson metric. We develop here tools that allow us to verify the small-gain condition in the examples treated in this paper. Due to space limitations, we do not cover the periodic case; for details on the latter, as well as proofs of properties of the Thompson metric used here, we refer the reader to Appendix D in [21].

In this section, and unless otherwise stated, by an (*algebraic*) *cone* we mean a nonempty subset $V_+ \subseteq V$ of a real vector space that satisfies $V_+ + V_+ \subseteq V_+$, $\alpha V_+ \subseteq V_+$ for every $\alpha > 0$, and $V_+ \cap (-V_+) \subseteq \{0\}$. Elements of V_+ are said to be *nonnegative*. Note that we do not impose any topology on V nor ask that V_+ be closed. Nevertheless, such a cone induces a partial order in the underlying vector space just like before by defining $x \leqslant y$ if and only if $y - x \in V_+$. This partial order is compatible with the linear structure of the vector space, in the sense that $x \leqslant y$ and $x' \leqslant y'$ imply $x + x' \leqslant y + y'$, and $tx \leqslant ty$ for every $t > 0$, and $x \geqslant y$ for every $t < 0$.

The equivalence classes under the equivalence relation,

$$\forall x, y \in V_+\,, \quad x \sim y \quad \Leftrightarrow \quad \exists c > 0: \quad c^{-1}x \leqslant y \leqslant cx,$$

are called the *parts* of V_+. In particular, $C_0 := \{0\}$ is a part whenever $0 \in V_+$; we refer to all other parts as the *nonzero parts of* V_+. For example, the only nonzero part of the cone $\mathbb{R}_{\geqslant 0} \subseteq \mathbb{R}$ is $\mathbb{R}_{>0}$, the nonzero parts of $\mathbb{R}^2_{\geqslant 0} \subseteq \mathbb{R}^2$ are $\{0\} \times \mathbb{R}_{>0}$, $\mathbb{R}_{>0} \times \{0\}$, and $\mathbb{R}_{>0} \times \mathbb{R}_{>0}$, and, in general, $\mathbb{R}^n_{\geqslant 0} \subseteq \mathbb{R}^n$ has $2^n - 1$ nonzero parts, namely, $\mathbb{R}^n_{>0}$ and the projections of $\mathbb{R}^n_{>0}$ over each of the lower-dimensional coordinate subspaces.

It is not hard to prove that if V_+ is a solid, closed cone in a normed space V, then $\operatorname{int} V_+$ is a part. If x, y, z are in the interior of V_+, then so are $x + z$ and $y + z$. In particular, $x, y, x + z, y + z$ are all in the same part of V_+.

For each nonzero part C of a cone V_+, the map $d_C\colon C \times C \to \mathbb{R}_{\geqslant 0}$, defined by

$$d_C(x, y) := \inf\{\log c\,;\ c^{-1}x \leqslant y \leqslant cx\}\,, \quad x, y \in C\,,$$

is called the *Thompson metric on* C. Unless there is any risk of ambiguity, we will omit the index "C" designating the part, writing simply "d" for the Thompson metric on any part. We set $d(0, 0) = 0$ by convention. Actually, in general, d_C is only a pseudometric. Sufficient conditions for d_C to be a metric are that V be a normed space in which the underlying cone V_+ is closed. This will be enough for our purposes in this paper. The reader interested in necessary and sufficient algebraic conditions for the Thompson metric to be an actual metric may consult [7] for a characterization in terms of the "almost Archimedean" property.

Thompson introduced d_C in [32], where he showed that, under the assumption that the underlying cone is normal, the metric is complete, and proved a fixed point result for a class of nonlinear operators which are contractive with respect to the metric. The Thompson metric is related to the *Hilbert projective metric*, a thorough account of which is given in [25, 26]. We summarize next a few needed properties. In all statements, V and W are real vector spaces partially ordered by cones $V_+ \subseteq V$ and $W_+ \subseteq W$, respectively.

A *sublinear* map $g\colon V_+ \to W_+$ is one for which $\lambda g(x) \leqslant g(\lambda x)$ for all $\lambda \in [0, 1]$ and $x \in V_+$. In particular, any linear $g^*\colon V \to W$ such that $g^*(V_+) \subseteq W_+$ is order-preserving and its restriction to V_+ is sublinear. Moreover, a composition $h \circ g\colon U_+ \to W_+$ of two sublinear maps $g\colon U_+ \to V_+$ and $h\colon V_+ \to W_+$ is sublinear, provided that h is also order-preserving.

Lemma 5.1. *If $g\colon V_+ \to W_+$ is order-preserving and sublinear, then g is nonexpansive with respect to the Thompson metric in the following sense: whenever x and y are in the same part of V_+, $g(x)$ and $g(y)$ are also in the same part of W_+, and $d(g(x), g(y)) \leqslant d(x, y)$.*

Lemma 5.2. *Given $\beta \in V_+$, let $\tau_\beta\colon V_+ \to V_+ : x \mapsto \beta + x$ be the translation of V_+ by β. Then τ_β is nonexpansive with respect to the Thompson metric.*

Proposition 5.3. *Suppose that V is a Banach space, partially ordered by a solid, closed cone $V_+ \subseteq V$. For any $\beta \in \operatorname{int} V_+$, the translation $\tau_\beta\colon \operatorname{int} V_+ \to \operatorname{int} V_+$ is nonexpansive with respect to the Thompson metric on $\operatorname{int} V_+$; that is, $d(\tau_\beta(x), \tau_\beta(y)) = d(x + \beta, y + \beta) \leqslant d(x, y)$ for all $x, y \in \operatorname{int} V_+$. Furthermore, for any $B \in \operatorname{int} V_+$, the* *restriction of τ_β to $[0,B] \cap \operatorname{int} V_+$ is a strict contraction; that is, there exists an $L = L(\beta, B) \in [0,1)$ such that $d(x+\beta, y+\beta) \leqslant Ld(x,y)$ for all $x, y \in \operatorname{int} V_+ \cap [0,B]$.*

PROPOSITION 5.4. *Let V be a real Banach space which is partially ordered by a cone $V_+ \subseteq V$. Then each of the parts of V_+ is complete with respect to the Thompson metric if and only if V_+ is normal.*

Lemmas 5.1 and 5.2 are not very difficult to prove. Proposition 5.3 follows from [20, Theorem 2.6, p. 85]. An elementary proof for Proposition 5.4 can be pieced together using well-known results from the theory of positive operators, as outlined in [21, p. 179].

We will deal, in particular, with cones of nonnegative functions, defined as follows for an arbitrary nonempty set T. Consider the space $(\mathbb{R}^k)^T$ of $\mathbb{R}^k$-valued functions on T. The positive orthant cone $\mathbb{R}^k_{\geqslant 0} \subseteq \mathbb{R}^k$ induces the cone $(\mathbb{R}^k)^T_+ := (\mathbb{R}^k_{\geqslant 0})^T$ of nonnegative functions in $(\mathbb{R}^k)^T$.

Given any $\alpha = (\alpha_1, \ldots, \alpha_k)$ and $u = (u_1, \ldots, u_k)$ in $(\mathbb{R}^k)^T$, the *Hadamard product* $\alpha \odot u$ of α and u, defined by $(\alpha \odot u)_j(t) := \alpha_j(t) u_j(t)$ for $t \in T$ and $j = 1, \ldots, k$, is bilinear and, in particular, $u \longmapsto \alpha \odot u$ is linear. If $\alpha \geqslant 0$, then this map is also order-preserving. For any $\alpha \in (\mathbb{R}^k_{>0})^T$, the *coordinatewise inverse* $\alpha^{-1} \colon T \to \mathbb{R}^k$ is defined as $\alpha^{-1}(t) := (1/\alpha_1(t), \ldots, 1/\alpha_k(t))$ for $t \in T$. For any u, v, and α in $(\mathbb{R}^k_{\geqslant 0})^T$ such that u and v are in the same part, $\alpha \odot u$ and $\alpha \odot v$ are also in the same part, and $d(\alpha \odot u, \alpha \odot v) \leqslant d(u,v)$; moreover, u^{-1} and v^{-1} are in the same part and $d(u^{-1}, v^{-1}) = d(u,v)$.

For the sake of convenience, we will refer to a measurable map $B \colon \mathbb{R} \to M_{n\times k}(\mathbb{R})$ as a *tempered path* if $K_\delta := \sup_{s\in\mathbb{R}} \|B(s)\| \,\mathrm{e}^{-\delta|s|} < \infty$ for each $\delta > 0$. In particular, B is locally essentially bounded. Note that the natural analogues of all properties of tempered random variables are still true for tempered paths. In particular, the family $L^\theta(M_{n\times k}(\mathbb{R}))$ of tempered paths $\mathbb{R} \to M_{n\times k}(\mathbb{R})$ is a vector space over the real scalars. Of course all the above also can be said about vector-valued paths $\mathbb{R} \to \mathbb{R}^n$ upon identifying $\mathbb{R}^n$ with $M_{n\times 1}(\mathbb{R})$. We equip $M_{n\times k}(\mathbb{R})$ with the partial order induced by the nonnegative orthant cone, that is, the cone of $n \times k$ real matrices having all their entries nonnegative. We then equip $L^\theta(M_{n\times k}(\mathbb{R}))$ with the partial order induced by the cone $L^\theta_+(M_{n\times k}(\mathbb{R}))$ of (Lebesgue-almost surely) nonnegative paths in $M_{n\times k}(\mathbb{R})$. A natural analogue of property (L2) for linear RDSI, for a locally integrable matrix path $A \colon \mathbb{R} \to M_{n\times n}(\mathbb{R})$, is as follows:

(L2′) there exist a $\lambda > 0$ and a tempered path $\gamma \colon \mathbb{R} \to \mathbb{R}$ such that the fundamental matrix solution $\Xi \colon \mathbb{R} \times \mathbb{R} \to M_{n\times n}(\mathbb{R})$ of the linear differential equation $\dot{\xi} = A(t)\xi$, $t \in \mathbb{R}$ satisfies $\|\Xi(s, s+r)\| \leqslant \gamma(s)\,\mathrm{e}^{-\lambda r}$ for all $s \in \mathbb{R}$ and all $r \geqslant 0$.

The proof of the following lemma uses Proposition 3.12 (Kamke condition), the sublinearity on V_+ of an order-preserving linear map, and Lemma 5.1.

LEMMA 5.5. *Let $A \colon \mathbb{R} \to M_{n\times n}(\mathbb{R})$ and $B \colon \mathbb{R} \to M_{n\times k}(\mathbb{R})$ be locally integrable matrix paths so that B is tempered and (L2′) holds, B is nonnegative (i.e., $B_{ij}(t) \geqslant 0$ for Lebesgue-almost every $t \in \mathbb{R}$, for $i = 1, \ldots, n$, $j = 1, \ldots, k$), and all off-diagonal entries of A are nonnegative; that is, $A_{ij}(t) \geqslant 0$ for Lebesgue-almost every $t \in \mathbb{R}$, for all $i, j = 1, \ldots, n$ such that $i \neq j$. Then*

$$[\mathcal{J}^*(u)](t) := \int_{-\infty}^{t} \Xi(\sigma, t) B(\sigma) u(\sigma)\, d\sigma\,, \quad t \in \mathbb{R}\,, \quad u \in L^\theta(\mathbb{R}^k)\,, \tag{5.1}$$

defines an order-preserving, linear operator $\mathcal{J}^* \colon L^\theta(\mathbb{R}^k) \to L^\theta(\mathbb{R}^n)$. In particular,

$$\mathcal{J} := \mathcal{J}^*\big|_{L^\theta_+(\mathbb{R}^k)} \colon L^\theta_+(\mathbb{R}^k) \to L^\theta_+(\mathbb{R}^n)$$

is sublinear and thus nonnexpansive with respect to the Thompson metric.

Conditions for strict contractiveness. We now consider the Banach space $L^\infty(\mathbb{R}^n)$ of Borel-measurable, essentially bounded, vector-valued functions $\mathbb{R} \to \mathbb{R}^n$, with the usual essential supremum norm, equipped with the partial order induced by the solid, normal cone $L^\infty_+(\mathbb{R}^n)$ of nonnegative (Borel-measurable and essentially bounded) functions $\mathbb{R} \to \mathbb{R}^n_{\geqslant 0}$. The interior $\operatorname{int} L^\infty_+(\mathbb{R}^n)$ of $L^\infty_+(\mathbb{R}^n)$ is the family of functions uniformly (essentially) bounded away from zero; that is, u belongs to $\operatorname{int} L^\infty_+$ if and only if there exists an $\epsilon = (\epsilon_1, \dots, \epsilon_n) \gg 0$ such that $\{t \in \mathbb{R};\ u(t) < \epsilon\}$ has Lebesgue measure zero. For any $u = (u_1, \dots, u_n) \in \operatorname{int} L^\infty_+$, $u^{-1} = (u_1^{-1}, \dots, u_n^{-1})$ is well-defined and also belongs to $\operatorname{int} L^\infty_+$. Any path $u \in L^\infty(\mathbb{R}^n)$ has a representative which is bounded everywhere. Assume without loss of generality that u is one such representative. Then indeed $u \in L^\theta(\mathbb{R}^n)$. Having this identification in mind, we may thus write $L^\infty(\mathbb{R}^n) \subseteq L^\theta(\mathbb{R}^n)$. The proofs of the following two results combine the previously stated properties of the Thompson metric, Lemma 5.5, Proposition 5.3 and Lemma 5.1. See [21] for details.

LEMMA 5.6. *Suppose that $A\colon \mathbb{R} \to M_{n\times n}(\mathbb{R})$ and $B\colon \mathbb{R} \to M_{n\times k}(\mathbb{R})$ are as in Lemma 5.5, and let $\mathcal{H}\colon L^\infty_+(\mathbb{R}^k) \longrightarrow L^\infty_+(\mathbb{R}^k)$ be defined by*

$$[\mathcal{H}(u)](t) := \left(\frac{\alpha_j(t)}{\beta_j(t) + g_j\big(\int_{-\infty}^t \Xi(\sigma,t)B(\sigma)u(\sigma)\,d\sigma\big)} \right)_{j=1}^k, \quad t \in \mathbb{R},$$

for each $u \in L^\infty_+(\mathbb{R}^k)$, where (i) *$\alpha = (\alpha_1, \dots, \alpha_k)$ and $\beta = (\beta_1, \dots, \beta_k)$ are in $\operatorname{int} L^\infty_+(\mathbb{R}^k) \cap C^0(\mathbb{R}^k)$, and* (ii) *$g = (g_1, \dots, g_k)\colon \mathbb{R}^n_{\geqslant 0} \to \mathbb{R}^k_{\geqslant 0}$ is continuous, bounded, order-preserving, and sublinear. Then $\mathcal{H}(L^\infty_+(\mathbb{R}^k)) \subseteq \operatorname{int} L^\infty_+(\mathbb{R}^k)$. Furthermore, the restriction $\mathcal{I}\colon \operatorname{int} L^\infty_+(\mathbb{R}^k) \longrightarrow \operatorname{int} L^\infty_+(\mathbb{R}^k)$ of $\mathcal{H}$ to $\operatorname{int} L^\infty_+(\mathbb{R}^k)$ is a strict contraction with respect to the Thompson metric on $\operatorname{int} L^\infty_+(\mathbb{R}^k)$; that is, there exists an $L \in [0,1)$ such that $d(\mathcal{I}(u), \mathcal{I}(v)) \leqslant L d(u,v)$ for all $u, v \in \operatorname{int} L^\infty_+(\mathbb{R}^k)$.*

Proof. Let $M_j > 0$ be such that $g_j(x) \leqslant M_j$ for every $x \in \mathbb{R}^k_{\geqslant 0}$, for $j = 1, \dots, k$. Define $M \in \operatorname{int} L^\infty_+(\mathbb{R}^k)$ by $M(t) := (M_1, \dots, M_k)$, $t \in \mathbb{R}$. By Proposition 5.3, there exists an $L := L(\beta/2, \beta/2 + M) \in [0,1)$ such that

$$d(\beta/2 + x, \beta/2 + y) \leqslant L d(x,y) \quad \forall x, y \in [\beta/2, \beta/2 + M] \cap \operatorname{int} L^\infty_+(\mathbb{R}^k)\,. \tag{5.2}$$

Fix $u \in L^\infty_+(\mathbb{R}^k)$ arbitrarily. It follows from (i) and (ii) that $\mathcal{H}(u)$ is nonnegative and bounded coordinatewise. Furthermore, $\alpha_j(t) \geqslant \epsilon_j$ and $\beta_j(t) \leqslant B_j$, for every $t \in \mathbb{R}$, for some $\epsilon_j > 0$ and $B_j < \infty$, for $j \in \{1, \dots, k\}$. Hence

$$\big([\mathcal{H}(u)](t)\big)_j \geqslant \frac{\epsilon_j}{B_j + M_j} > 0 \quad \forall t \in \mathbb{R}, \quad \forall j \in \{1, \dots, k\}\,.$$

This shows that $\mathcal{H}(L^\infty_+(\mathbb{R}^k)) \subseteq \operatorname{int} L^\infty_+(\mathbb{R}^k)$.

To establish the strict contractiveness part of the result, consider the operator $\mathcal{G}\colon L^\infty_+(\mathbb{R}^n) \to L^\infty_+(\mathbb{R}^k)$, defined by

$$[\mathcal{G}(\xi)](t) := g(\xi(t)) = (g_1(\xi(t)), \dots, g_k(\xi(t)))\,, \quad t \in \mathbb{R}, \quad \xi \in L^\infty_+(\mathbb{R}^n)\,.$$

Note that $\mathcal{G}$ is also sublinear and order-preserving. Combining this with the various pieces of notation introduced above, we may rewrite

$$\mathcal{H}(u) = \alpha \odot \big(\beta + \mathcal{G}\big(\mathcal{J}(u)\big)\big)^{-1} \quad \forall u \in \operatorname{int} L^\infty_+(\mathbb{R}^k)\,.$$

Fix arbitrarily $u, v \in \operatorname{int} L_+^\infty(\mathbb{R}^k)$. We have

$$\begin{aligned} d(\mathcal{I}(u), \mathcal{I}(v)) &\leqslant d\left(\left(\beta + \mathcal{G}\big(\mathcal{J}(u)\big)\right)^{-1}, \left(\beta + \mathcal{G}\big(\mathcal{J}(v)\big)\right)^{-1}\right) \\ &= d\left(\beta + \mathcal{G}\big(\mathcal{J}(u)\big), \beta + \mathcal{G}\big(\mathcal{J}(v)\big)\right). \end{aligned}$$

Since $\mathcal{G}(L_+^\infty(\mathbb{R}^n)) \subseteq [0, M]$, it follows from (5.2) that

$$d\left(\beta + \mathcal{G}\big(\mathcal{J}(u)\big), \beta + \mathcal{G}\big(\mathcal{J}(v)\big)\right) \leqslant L d\left(\beta/2 + \mathcal{G}\big(\mathcal{J}(u)\big), \beta/2 + \mathcal{G}\big(\mathcal{J}(v)\big)\right).$$

Hence

$$\begin{aligned} d(\mathcal{I}(u), \mathcal{I}(v)) &\leqslant L d\left(\beta/2 + \mathcal{G}\big(\mathcal{J}(u)\big), \beta/2 + \mathcal{G}\big(\mathcal{J}(v)\big)\right) \\ &\leqslant d(u, v), \end{aligned}$$

completing the proof. ∎

Reasoning along the same lines, one may obtain the following.

LEMMA 5.7. *Assume the same hypotheses as in Lemma* 5.6*, except for replacing* (i) *and* (ii) *in that lemma by* (i′) $\alpha = (\alpha_1, \ldots, \alpha_k)$, $\widetilde{\alpha} = (\widetilde{\alpha}_1, \ldots, \widetilde{\alpha}_k)$, $\beta = (\beta_1, \ldots, \beta_k)$ *and* $\widetilde{\beta} = (\widetilde{\beta}_1, \ldots, \widetilde{\beta}_k)$ *are in* $\operatorname{int} L_+^\infty(\mathbb{R}^k) \cap C^0(\mathbb{R}^k)$*, and satisfy* $\alpha_j(t)/\beta_j(t) \geqslant \widetilde{\alpha}_j(t)/\widetilde{\beta}_j(t)$ *for all* $t \in \mathbb{R}$, $j = 1, \ldots, k$ *and* (ii′) $g = (g_1, \ldots, g_k)\colon \mathbb{R}^n_{\geqslant 0} \to \mathbb{R}^k_{\geqslant 0}$ *is continuous, order-preserving and sublinear. Let* $\mathcal{H}\colon L_+^\infty(\mathbb{R}^k) \to L_+^\infty(\mathbb{R}^k)$ *be defined by*

$$[\mathcal{H}(u)](t) := \left(\frac{\alpha_j(t) + \widetilde{\alpha}_j(t) g_j\left(\int_{-\infty}^t \Xi(\sigma, t) B(\sigma) u(\sigma)\, d\sigma\right)}{\beta_j(t) + \widetilde{\beta}_j(t) g_j\left(\int_{-\infty}^t \Xi(\sigma, t) B(\sigma) u(\sigma)\, d\sigma\right)} \right)_{j=1}^k, \quad t \in \mathbb{R},$$

for each $u \in L_+^\infty(\mathbb{R}^k)$*. Then the same conclusions as in Lemma* 5.6 *hold.*

We can now combine the previous results in order to provide a result on uniqueness and global attraction for the discrete iteration in the small-gain theorem.

LEMMA 5.8. *Under the same hypotheses as in either Lemma* 5.6 *or Lemma* 5.7*, the discrete dynamical system on* $\operatorname{int} L_+^\infty(\mathbb{R}^k)$ *generated by the difference equation*

$$u^+ = \mathcal{H}(u) \tag{5.3}$$

has a unique, globally attracting equilibrium $u_\infty \in \operatorname{int} L_+^\infty(\mathbb{R}^k)$ *with respect to the Thompson metric on the part* $\operatorname{int} L_+^\infty(\mathbb{R}^k)$.

Proof. We already remarked that $\operatorname{int} L_+^\infty(\mathbb{R}^k)$ is a part. Since $L^\infty(\mathbb{R}^k)$ is a Banach space and $L_+^\infty(\mathbb{R}^k)$ is a normal cone, the Thompson metric on $\operatorname{int} L_+^\infty(\mathbb{R}^k)$ is complete by Proposition 5.4. Under the hypotheses of either Lemma 5.6 or Lemma 5.7,

$$\mathcal{H}(\operatorname{int} L^\infty(\mathbb{R}^k)) \subseteq \mathcal{H}(L^\infty(\mathbb{R}^k)) \subseteq \operatorname{int} L^\infty(\mathbb{R}^k),$$

and $\mathcal{H}$ is a strict contraction (with respect to the Thompson metric) on $\operatorname{int} L_+^\infty(\mathbb{R}^k)$. Therefore $\mathcal{H}$ has a unique, globally attracting fixed point $u_\infty \in \operatorname{int} L^\infty(\mathbb{R}^k)$ by the Banach fixed point theorem. ∎

PROPOSITION 5.9. *Suppose that* $A\colon \mathbb{R} \to M_{n\times n}(\mathbb{R})$ *and* $B\colon \mathbb{R} \to M_{n\times k}(\mathbb{R})$ *are as in Lemma* 5.5*, and let* $\mathcal{H}\colon L_+^\theta(\mathbb{R}^k) \longrightarrow L_+^\theta(\mathbb{R}^k)$ *be defined in the same manner as in Lemma* 5.6*, assuming* (i) *and* (ii)*, but now with inputs* $u \in L_+^\theta(\mathbb{R}^k)$*. Then the discrete dynamical system on* $L_+^\theta(\mathbb{R}^k)$ *generated by the difference equation*

$$u^+ = \mathcal{H}(u)$$

has a unique, globally attracting equilibrium $u_\infty \in \operatorname{int} L^\infty_+(\mathbb{R}^k) \subseteq L^\theta_+(\mathbb{R}^k)$ *in the sense of almost-everywhere, pointwise convergence. Furthermore, the representative* u_∞ *can be chosen to be continuous, in which case convergence is actually everywhere; that is,*

$$[\mathcal{H}^k(u)](t) \longrightarrow u_\infty(t) \quad \textit{as} \quad k \to \infty \quad \forall t \in \mathbb{R}\,, \quad \forall u \in L^\theta_+(\mathbb{R}^k)\,.$$

Proof. The assumptions imply that $\mathcal{H}(u)$ is in fact bounded coordinatewise for each $u \in L^\theta_+(\mathbb{R}^k)$, and therefore $\mathcal{H}(L^\theta_+(\mathbb{R}^k)) \subseteq L^\infty_+(\mathbb{R}^k) \subseteq L^\theta(\mathbb{R}^k)$. By Lemma 5.6, we also have $\mathcal{H}(L^\infty_+(\mathbb{R}^k)) \subseteq \operatorname{int} L^\infty_+(\mathbb{R}^k)$, so also $\mathcal{H}^2(L^\theta_+(\mathbb{R}^k)) \subseteq \operatorname{int} L^\infty_+(\mathbb{R}^k)$. By Lemma 5.8, $\mathcal{H}\big|_{\operatorname{int} L^\infty_+(\mathbb{R}^k)}$ has a unique, globally attracting fixed point u_∞ with respect to the Thompson metric. Now fix $u \in L^\theta_+(\mathbb{R}^k)$ arbitrarily, and let $u_k := \mathcal{H}^k(u)$, $k = 0, 1, 2, \ldots$. Then $u_k \in \operatorname{int} L^\infty_+(\mathbb{R}^k)$, $k = 2, 3, 4, \ldots$. Moreover, $d(u_k, u_\infty) \longrightarrow 0$ as $k \to \infty$, since u_∞ is the unique, globally attracting fixed point of $\mathcal{H}\big|_{\operatorname{int} L^\infty_+(\mathbb{R}^k)}$ with respect to the Thompson metric. Now $\mathrm{e}^{-d(u_k,u_\infty)}\, u_\infty \leqslant u_k \leqslant \mathrm{e}^{d(u_k,u_\infty)}\, u_\infty$, $k = 2, 3, 4, \ldots$. Thus by the triangle inequality, and by normality,

$$\begin{aligned}
\|u_k - u_\infty\|_\infty &\leqslant \|u_k - \mathrm{e}^{-d(u_k,u_\infty)}\, u_\infty\|_\infty + \|\,\mathrm{e}^{-d(u_k,u_\infty)}\, u_\infty - u_\infty\|_\infty \\
&\leqslant 1 \cdot \|(\mathrm{e}^{d(u_k,u_\infty)} - \mathrm{e}^{-d(u_k,u_\infty)})u_\infty\|_\infty + \|(\mathrm{e}^{-d(u_k,u_\infty)} - 1)u_\infty\|_\infty \\
&\leqslant (|\,\mathrm{e}^{d(u_k,u_\infty)} - \mathrm{e}^{-d(u_k,u_\infty)}\,| + |\,\mathrm{e}^{-d(u_k,u_\infty)} - 1|)\|u\|_\infty \\
&\to 0 \quad \text{as} \quad k \to \infty\,.
\end{aligned}$$

In particular, $(u_k)_{k\in\mathbb{N}}$ is a Cauchy sequence. Furthermore, u_k is continuous for each $k \in \mathbb{N}$, since α, β, g are continuous by hypothesis and $\mathcal{J}(u)$ is continuous for each $u \in L^\theta_+(\mathbb{R})$. Thus indeed $(u_k)_{k\in\mathbb{N}}$ converges uniformly to a continuous function which is equal to u_∞ in the sense of L^∞. ∎

A totally analogous proof, but now appealing to Lemma 5.7, gives the following.

PROPOSITION 5.10. *Suppose that* $A\colon \mathbb{R} \to M_{n\times n}(\mathbb{R})$ *and* $B\colon \mathbb{R} \to M_{n\times k}(\mathbb{R})$ *are as in Lemma* 5.5*, and let* $\mathcal{H}\colon L^\theta_+(\mathbb{R}^k) \longrightarrow L^\theta_+(\mathbb{R}^k)$ *be defined in the same manner as in Lemma* 5.7*, assuming* (i′) *and* (ii′)*, but now with inputs* $u \in L^\theta_+(\mathbb{R}^k)$*. Then the discrete dynamical system on* $L^\theta_+(\mathbb{R}^k)$ *generated by the difference equation*

$$u^+ = \mathcal{H}(u)$$

has a unique, globally attracting equilibrium $u_\infty \in \operatorname{int} L^\infty_+(\mathbb{R}^k) \subseteq L^\theta_+(\mathbb{R}^k)$ *in the sense of almost-everywhere, pointwise convergence. Furthermore, the representative* u_∞ *can be chosen to be continuous, in which case convergence is actually everywhere; that is,*

$$[\mathcal{H}^k(u)](t) \longrightarrow u_\infty(t) \quad \textit{as} \quad k \to \infty \quad \forall t \in \mathbb{R}\,, \quad \forall u \in L^\theta_+(\mathbb{R}^k)\,.$$

6. Discussion/closing remarks. We have developed the foundations for an extension of Arnold's RDS formalism to encompass systems with inputs and outputs. The usefulness of our approach was illustrated by the statement and proof of a small-gain theorem for RDS. We view this theorem as merely a first step in the development of a theory of RDSIO. Future directions to pursue include the study of more complicated network interconnections, SDE (with inputs and outputs), as well as, for example, the development of an extension to RDSIO of notions of input-to-state stability and the associated methods for nonlinear systems analysis and control design.

Appendix A. BMNSO spaces. In this appendix we carefully define BMNSO spaces, presenting some technical properties which were needed throughout this paper.

Spaces with the exact same structure but without any special name have been called for in the work of Chueshov on monotone RDS [6]. Our main goal in this appendix is, therefore, to make the presentation more self-contained by collecting all necessary definitions and providing primary references for results which are not proved here. It also presents a few crucial technical results and helpful terminology not found in [6].

A.1. The Hausdorff distance. Recall that the *Hausdorff distance* between two nonempty, bounded subsets A and C of a metric space (X, d) is defined to be the nonnegative real number

$$d_H(A, C) := \max\left\{\sup_{a\in A} \operatorname{dist}(a, C),\ \sup_{c\in C} \operatorname{dist}(c, A)\right\},$$

where $\operatorname{dist}(x, E) := \inf_{y\in E} d(x, y)$ for $x \in X$ and $\varnothing \neq E \subseteq X$ is the distance between a point and a nonempty subset of X. Given a point $x \in X$, and $\epsilon > 0$, denote by $B_\epsilon(x)$ the *ball of radius* ϵ *and centered at* x, that is, $B_\epsilon(x) := \{y \in X\,;\ d(y, x) < \epsilon\}$, and let $A_\epsilon := \bigcup_{a\in A} B_\epsilon(a)$, for any nonempty subset $A \subseteq X$. It is not difficult to show that $d_H(A, C) = \inf\{\epsilon > 0\,;\ A \subseteq C_\epsilon \text{ and } C \subseteq A_\epsilon\}$ for any nonempty, bounded subsets $A, C \subseteq X$.

Given a metric space (X, d), we denote the family of nonempty, bounded, closed subsets of X by $F(X)$. It is well-known that when (X, d) is a compact metric space, the restriction $d_H\big|_{F(X)\times F(X)}$ of the Hausdorff distance to $F(X)$ constitutes a metric with respect to which $F(X)$ is also compact (see, e.g., [21, Appendix A] for a self-contained presentation). This property of the Hausdorff distance will be used below to show that the shell of a compact subset of an BMNSO space is also compact (Theorem A.6).

A.2. BMNSO spaces. Recall that a subset V_+ of a real topological vector space V is said to be a *cone* if (C1) V_+ is closed (not typically part of the definition [20], but a standard assumption in the theory of monotone RDS [6, 5], and also needed in our preliminary results); (C2) $V_+ + V_+ := \{x + y\,;\ x, y \in V_+\} \subseteq V_+$; (C3) $\alpha V_+ := \{\alpha x\,;\ x \in V_+\} \subseteq V_+$ for every $\alpha \geqslant 0$; and (C4) $V_+ \cap (-V_+) = \{0\}$. Given a subset X of V and a cone $V_+ \subseteq V$, the binary relation $\leqslant_{V_+}$ on X defined by $x \leqslant_{V_+} y \Leftrightarrow y - x \in V_+$ is a *closed* partial order, that is, $\{(x, y) \in X \times X\,;\ x \leqslant_{V_+} y\}$ is a closed subset of $X \times X$. This partial order is referred to as the *partial order in* X *induced by the cone* V_+. Naturally, we write $x <_{V_+} y$ to indicate that $x \leqslant_{V_+} y$ and $x \neq y$. Furthermore, $x \geqslant_{V_+} y$ means that $y \leqslant_{V_+} x$, and $x >_{V_+} y$ means that $y <_{V_+} x$.

An *order-interval* of V is a subset of the form $\{x \in V\,;\ aR_1xR_2b\}$, $\{x \in V\,;\ aR_1x\}$, or $\{x \in V\,;\ xR_2b\}$, for some $R_1, R_2 \in \{\leqslant_{V_+}, <_{V_+}\}$ and $a, b \in V$. We denote, in particular, $[a, b] := \{x \in V\,;\ a \leqslant_{V_+} x \leqslant_{V_+} b\}$, $a, b \in V$.

If the interior $\operatorname{int} V_+$ of the cone V_+ is nonempty, then V_+ is said to be a *solid* cone. In this case we write $x \ll_{V_+} y$ or $y \gg_{V_+} x$ whenever $y - x \in \operatorname{int} V_+$.

The index V_+ in the inequality symbols above shall be dropped whenever there is no risk of confusion regarding the underlying cone.

A vector $v \in V$ is said to be a *supremum* of a subset $A \subseteq V$—and denoted by $\sup A$—if $a \leqslant v$ for every $a \in A$ (i.e., v is an *upper bound*), and $v \leqslant \tilde{v}$ for any $\tilde{v} \in V$ such that $a \leqslant \tilde{v}$ for every $a \in A$ (the *least upper bound*). Note that the supremum, if it exists, is unique. *Lower bounds* and *infima* are defined analogously.

If every order-bounded, finite subset $M = \{v_1, \dots, v_k\} \subseteq V$ has a supremum $\sup M$, then the cone V_+ is said to be *minihedral*.

For ease of reference, we state the property below as a lemma. The proof illustrates the kind of situation in which it is convenient to assume that the cone is a closed subset of its underlying topological vector space.

LEMMA A.1. *Let B be a subset of a real topological vector space partially ordered by a cone. Then* $\sup B$ *exists if and only if* $\sup \overline{B}$ *exists. In this case,* $\sup B = \sup \overline{B}$. *Analogously,* $\inf B$ *exists if and only if* $\inf \overline{B}$ *exists, in which case* $\inf B = \inf \overline{B}$.

Proof. Suppose $\sup B$ exists. Given any $x \in \overline{B}$, let $(x_\alpha)_{\alpha\in A}$ be a net in B converging to x. We have $x_\alpha \leqslant \sup B$ for all $\alpha \in A$, hence $x \leqslant \sup B$ by taking limits on both sides of the inequality. Thus $\sup B$ is an upper bound for $\overline{B}$. Now suppose $v \in X$ is any upper bound for $\overline{B}$. Then v is also an upper bound for B, and so $\sup B \leqslant v$ by the definition of suprema for B. This shows that $\sup \overline{B}$ exists and is equal to $\sup B$.

Conversely, if $\sup \overline{B}$ exists, then it is clearly an upper bound for B. Furthermore, any upper bound for B is also an upper bound for $\overline{B}$, as we saw above, and thus greater than or equal to $\sup \overline{B}$. This proves that $\sup B$ exists and is equal to $\sup \overline{B}$.

The proof for infima is entirely analogous. ∎

Now suppose that V is a normed vector space. Then V_+ is said to be *normal* if there exists a constant $C_{V_+} \geqslant 0$ such that $0 \leqslant x \leqslant y$ implies $\|x\| \leqslant C_{V_+}\|y\|$.

LEMMA A.2. *Suppose that $(x_\alpha)_{\alpha\in A}$ is a net in a normed space V, partially ordered by a solid, normal cone $V_+ \subseteq V$. Suppose, in addition, that the net converges to an element $x_\infty \in V$ and that the infima and suprema*

$$x_\alpha^- := \inf\{x_{\alpha'}\,;\ \alpha' \geqslant \alpha\} \quad \textit{and} \quad x_\alpha^+ := \sup\{x_{\alpha'}\,;\ \alpha' \geqslant \alpha\}$$

exist for every $\alpha \in A$. Then the nets $(x_\alpha^-)_{\alpha\in A}$ and $(x_\alpha^+)_{\alpha\in A}$ so defined also converge to x_∞.

Proof. Since the cone V_+ is solid, we may choose an u in the interior of V_+ such that the order-interval $[-u, u]$ contains the unit ball $B_1(0)$. Then

$$B_r(x_\infty) \subseteq [x_\infty - ru, x_\infty + ru] \quad \forall r > 0\,.$$

So, from the hypothesis of convergence, for each $r > 0$, there exists an $\alpha_r \in A$ such that $x_\alpha \in B_r(x_\infty) \subseteq [x_\infty - ru, x_\infty + ru]$ for all $\alpha \geqslant \alpha_r$. Now

$$x_\alpha^- = \inf\{x_{\alpha'}\,;\ \alpha' \geqslant \alpha\} \geqslant x_\infty - ru \quad \forall \alpha \geqslant \alpha_r\,, \quad \forall r > 0\,,$$

and, similarly,

$$x_\alpha^+ = \sup\{x_{\alpha'}\,;\ \alpha' \geqslant \alpha\} \leqslant x_\infty + ru \quad \forall \alpha \geqslant \alpha_r\,, \quad \forall r > 0\,;$$

that is,

$$x_\infty - ru \leqslant x_\alpha^- \leqslant x_\alpha^+ \leqslant x_\infty + ru \quad \forall \alpha \geqslant \alpha_r\,, \quad \forall r > 0\,.$$

Let $C_{V_+} \geqslant 0$ be the normality constant of V_+. Then

$$\begin{aligned}\|x_\alpha^- - x_\infty\| &\leqslant \|x_\alpha^- - (x_\infty - ru)\| + \|ru\| \\ &\leqslant C_{V_+}\|(x_\infty + ru) - (x_\infty - ru)\| + r\|u\| \\ &= (2C_{V_+} + 1)\|u\|r \quad \forall \alpha \geqslant \alpha_r\,, \quad \forall r > 0\,.\end{aligned}$$

Since $(2C_{V_+} + 1)\|u\|r \longrightarrow 0$ as $r \to 0$, we conclude that $\|x_\alpha^- - x_\infty\| \longrightarrow 0$. The proof that $\|x_\alpha^+ - x_\infty\| \longrightarrow 0$ as well is entirely analogous. ∎

DEFINITION A.3 (BMNSO spaces). *A real Banach space V which is partially ordered by a solid, normal, minihedral cone $V_+ \subseteq V$ shall be referred to as an* BMNSO *space.*

The remainder of this section is devoted to the review and development of a few key properties of BMNSO spaces. Definition A.5 and Theorem A.6 are key to the concepts of limit inferior and limit superior developed in subsection 2.2.

PROPOSITION A.4. *Suppose that V is an BMNSO space. Then every precompact subset $B \subseteq V$ has a supremum and an infimum. In particular,* $\sup B = \sup \overline{B}$ *and* $\inf B = \inf \overline{B}$.

Proof. For compact $B \subseteq V$, see [17, Theorem 6.5, p. 62]. It follows for precompact sets in view of Lemma A.1 that if B is precompact, then $\overline{B}$ is compact and so $\sup B = \sup \overline{B}$ and $\inf B = \inf \overline{B}$ by the lemma. ◻

In view of this proposition, the definition below is well-posed.

DEFINITION A.5 (shells). *For any compact subset K of a BMNSO space, the set* $\mathrm{shell}(K) := \{\sup E\,;\ E$ *is a precompact subset of* $K\}$ *is called the* shell *of* K.

THEOREM A.6. *The shell of a compact subset of a BMNSO space is compact.*

Proof. Let X be an arbitrary compact subset of an arbitrary BMNSO space V. By Lemma A.1, we have $\mathrm{shell}(X) = \{\sup E\,;\ E \in F(X)\}$, where $F(X)$ is the family of compact subsets of X. Now $F(X)$ is a compact metric space with respect to the Hausdorff distance d_H, and $\mathrm{shell}(X)$ is the image of $F(X)$ under the map

$$E \longmapsto \sup E\,, \quad E \in F(X)\,. \tag{A.1}$$

Therefore to prove the theorem it is enough to show that this map is continuous.

Let $u \in \mathrm{int}\, V_+$ be such that $B_1(0) \subseteq [-u, u]$, as in the proof of Lemma A.2. Denote the normality constant of V_+ by K. Fix arbitrarily $\delta > 0$ and $A, C \in F(X)$ such that $d_H(A, C) < \delta$. We have $A \subseteq C_\delta = C + B_\delta(0) \subseteq C + [-\delta u, \delta u]$, hence $\sup A \leqslant \sup C + \delta u$. Similarly, we can show that $\sup C \leqslant \sup A + \delta u$. Combining these two inequalities, we obtain $0 \leqslant \sup A - \sup C + \delta u \leqslant 2\delta u$. Now, by normality,

$$\|\sup A - \sup C\| \;\leqslant\; \|\sup A - \sup C + \delta u\| + \delta\|u\| \;\leqslant\; (2K+1)\|u\|\delta\,.$$

This shows that (A.1) is in fact uniformly continuous on $F(X)$, completing the proof. ◻

Appendix B. List of abbreviations.

BMNSO (space)	Banach space partially ordered by a solid, normal, minihedral cone.
CICS	convergent-input-to-convergent-state.
I/O (characteristic)	input-to-output characteristic.
I/S (characteristic)	input-to-state characteristic.
MPDS	measure-preserving dynamical system.
RDE	random differential equation.
RDEI	random differential equation with inputs.
RdEI	random difference equation with inputs.
RDS	random dynamical system.
RDSI	random dynamical system with inputs.
RDSIO	random dynamical system with inputs and outputs.

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