A Notion of Input to Output Stability

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Abstract. This paper deals with a notion of "input to output stability" (IOS), which formalizes the idea that outputs depend in an "aymptotically stable" manner on inputs, while internal signals remain bounded. When the output equals the complete state, one recovers the property of input to state stability (ISS). When there are no inputs, one has a generalization of the classical concept of partial stability. The main results provide Lyapunov-function characterizations of IOS.

Keywords. Stability, Nonlinear control, Robust stability

1 Introduction

This paper concerns itself with questions of stability for general finite-dimensional systems, in the standard sense of nonlinear control:

$$\dot{x}(t) = f(x(t), u(t)), \quad y(t) = h(x(t)) \tag{1}$$

(dot indicates derivative, and we often omit the time argument "t"), whose states $x(t)$ evolve in an Euclidean space $\mathbb{R}^n$. For the purposes of the present paper, it is useful to think of the possible forcing input functions $u(\cdot)$ as "disturbances" acting on the system, rather than controls to be manipulated, and to think of the output variable $y(t)$ as a quantity to be regulated, like a tracking error at time t. (Technical assumptions on f, h, and admissible inputs, are described later.) We wish to emphasize that *the results to be presented are new and of interest* — though easier — *even when there are no inputs* ($\dot{x} = f(x)$), a situation which we view as a special case of (1).

In many problems, it is usually the case that one only wishes to stabilize the output values $y(t)$ rather than the full state $x(t)$. Typically, it is required that $y(t)$ converge to zero as $t \to \infty$, and in addition one asks that internal variables $x(t)$ remain bounded (under suitable assumptions on the "disturbances" u). A very special case of this kind of question has a long and distinguished history in differential equation theory. Indeed, when there are no inputs u, and the coordinates of y are a subset of the coordinates of x (that is to say, h is a projection on a subspace of the state space $\mathbb{R}^n$), the type of property being considered is a concept of "partial" asymptotic stability; the reader is directed to the excellent survey paper [16] for references to partial stability, and to [6] for the somewhat related notion of "stability with respect to two measures".

There are several ways of making mathematically precise the objective described in the previous paragraph, and these alternatives vary in subtle details. For instance, one must decide how uniform is the rate of convergence of $y(t)$ to zero, and precisely how the magnitude of inputs and initial states affect this convergence. It is imperative to understand which of the possible formulations give rise to a theory which is both mathematically rich and has applied relevance for control theory. That is the main objective of this paper. One general guideline that we follow is that a mathematically natural notion should admit a characterization in Lyapunov-function terms.

In past work, we have studied a property which we call "input to state stability" (ISS, for short), introduced in [9], and have provided several characterizations, including a necessary and sufficient Lyapunov-theoretic one; for references see [12], [11], and the latest paper [13]. (For applications and more discussion of that concept see, for instance, [3], [4], [5], [15].) In very informal terms, the ISS property means that "no matter what the initial conditions, if the future inputs are small, then the state must eventually be small". In this work, we return to a subject also introduced in [9]. There, we used the term *input to output stability* (IOS, for short) to mean that the output (as opposed to the full state) must be eventually small, no matter what the initial conditions, if future inputs are small. It is possible to express the property in purely input/output terms, using past inputs to represent initial conditions, or in state space terms, explicitly summarizing the effect of past inputs through an initial state. Since we wish to provide Lyapunov-theoretic characterizations, we adopt this latter point of view here. (The relations between both views are explained in [9] and in more detail in [7] and [4].)

It turns out, however, that the notion of IOS given in [9] is *not* the appropriate one for modeling the situation typical in regulation or in robust and adaptive control, where a condition of *boundedness of inter-*

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nal variables is required in addition to asking that outputs become small. An equally important shortcoming, from a mathematical point of view, is that it would appear to be difficult to obtain a Lyapunov-theoretic characterization unless one imposes such internal boundedness. Thus, in this paper, we incorporate a state bound. For lack of a less cumbersome name, we decided to use the same term "IOS" for this new concept. This should not cause much confusion since the term has not been widely accepted.

We caution the reader not to confuse IOS with the notion named *input/output to state stability* (IOSS) in [14] (also called "detectability" in [10], and "strong unboundedness observability" in [4]). This other notion roughly means that "no matter what the initial conditions, if future inputs *and* outputs are small, the state must be eventually small". It is not a notion of stability; for instance, the unstable system $\dot{x} = x, y = x$ is IOSS. Rather, it represents a property of zero-state detectability. There is a fairly obvious connection between the various concepts introduced, however: a system is ISS if and only if it is both IOSS and IOS. This fact generalizes the linear systems theory result "internal stability is equivalent to detectability plus external stability" and its proof follows by routine arguments ([9], [7], [4]).

The next section presents precise definitions and statements of results. An appendix sketches the most important steps; a journal paper will contain complete proofs, which are fairly technical and long.

2 Statements of Results

We assume, for the systems (1) being considered, that the map $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is locally Lipschitz continuous, and the map $h : \mathbb{R}^n \to \mathbb{R}^p$ is continuous. We also assume that $f(0,0) = 0$ and $h(0) = 0$. We use the symbol $|\cdot|$ for Euclidean norm in $\mathbb{R}^n$, $\mathbb{R}^m$, and $\mathbb{R}^p$.

By an *input* we mean a measurable and locally essentially bounded function $u : \mathcal{I} \to \mathbb{R}^m$, where $\mathcal{I}$ is a subinterval of $\mathbb{R}$ which contains the origin. The L^m_∞-norm (possibly infinite) of an input u is denoted by $\|u\|$, i.e. $\|u\| = (\text{ess})\sup\{|u(t)|, t \in \mathcal{I}\}$. Whenever the domain $\mathcal{I}$ of an input u is not specified, it will be understood that $\mathcal{I} = \mathbb{R}_{\geq 0}$.

Given a system with control-value set $\mathbb{R}^m$, we often consider the same system but with controls restricted to take values in some subset $\Omega \subseteq \mathbb{R}^m$; we use $\mathcal{M}_\Omega$ for the set of all such controls.

Given any input u defined on an interval $\mathcal{I}$ containing $t = 0$, and given any $\xi \in \mathbb{R}^n$, there is a unique maximal solution of the initial value problem $\dot{x} = f(x,u)$, $x(0) = \xi$. This solution is defined on some maximal open subinterval of $\mathcal{I}$, and it is denoted by $x(\cdot,\xi,u)$. The corresponding output is denoted by $y(\cdot,\xi,u)$, that is, $y(t,\xi,u) = h(x(t,\xi,u))$ on the domain of definition of the solution.

As usual, we let $\mathcal{K}$ be the class of functions $[0,\infty) \to [0,\infty)$ which are zero at zero, strictly increasing, and continuous, $\mathcal{K}_\infty$ the subset of $\mathcal{K}$ functions that are unbounded, and $\mathcal{KL}$ the class of functions $[0,\infty)^2 \to [0,\infty)$ which are of class $\mathcal{K}$ on the first argument and decrease to zero on the second argument.

2.1 Main Concepts

Definition 2.1 A system (1) is *uniformly bounded input bounded state* (UBIBS) if there exists some $\mathcal{K}$-function σ such that, for every input u and every initial state ξ, the solution $x(t,\xi,u)$ is defined for all $t \geq 0$ and the estimate

$$|x(t,\xi,u)| \leq \max\{\sigma(|\xi|),\ \sigma(\|u\|)\}, \qquad \forall\, t \geq 0\,. \quad (2)$$

holds. □

Remark 2.2 The term UBIBS is sometimes employed for a weaker property, in which an additive constant is allowed in the right-hand side of the estimate (2), that is, one does not ask that trajectories must remain small if initial states and controls are small; see for instance [1]. As with other stability concepts (cf. the last section of [13]), much of what we do can be also stated in this weaker sense of "practical stability". □

Note that for a system with no controls,

$$\dot{x} = f(x),\ y = h(x), \quad (3)$$

the UBIBS property reduces to a "bounded state" estimate:

$$|x(t,\xi)| \leq \sigma(|\xi|),\ \forall\, t \geq 0, \forall\, \xi \in \mathbb{R}^n \quad (4)$$

This property amounts to (neutral) stability plus (cf. [17]) "uniform boundedness".

The main property that we wish to introduce is as follows.

Definition 2.3 A system (1) is *input to output stable* (IOS) if:

- it is UBIBS, and
- there exist a $\mathcal{KL}$-function and a $\mathcal{K}$-function γ such that

$$|y(t,\xi,u)| \leq \beta(|\xi|, t) + \gamma(\|u\|), \qquad \forall t \geq 0, \quad (5)$$

holds for all u and all $\xi \in \mathbb{R}^n$. □

For an autonomous system (3), we say simply *output stable* (OS). That is, such a system is OS if it satisfies an estimate (4) and there is some $\mathcal{KL}$-function β such that

$$|y(t,\xi)| \leq \beta(|\xi|, t) \qquad \forall t \geq 0, \quad (6)$$

holds for all $\xi \in \mathbb{R}^n$. Clearly, if system (1) is IOS then the associated 0-input system $\dot{x} = f(x,0)$ is OS.

When h is the identity, IOS coincides with the ISS property mentioned in the introductory section (and OS is exactly the same as global asymptotic stability). Observe that the UBIBS property is redundant in that case, as it is obviously implied by the decay estimate (5), letting $\sigma = \min\{\beta(\cdot,0),\gamma\}$.

Our main result is as follows.

Theorem 1 *The following statements are equivalent.*

1. *System (1) is* IOS.
2. *The system is* UBIBS *and there exist* $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, $\chi \in \mathcal{K}$ *and* $\alpha_3 \in \mathcal{KL}$ *such that for each* $r \geq 0$, *there is a smooth function* V *so that:*
   - $\alpha_1(|h(\xi)|) \leq V(\xi) \leq \alpha_2(|\xi|)$ *for all* $\xi \in \mathbb{R}^n$*;*
   - $DV(\xi)f(\xi,\mu) \leq -\alpha_3(V(\xi), |\xi|)$ *for all* ξ *such that* $V(\xi) \geq \chi(r)$ *and all* $|\mu| \leq r$.

The statement of the Lyapunov condition in Theorem (1) is more complicated than one would like. It says in essence that IOS is equivalent, for each bound on controls, to the existence of an Lyapunov-like function V which vanishes only when the output vanishes, and whose derivative along trajectories is negative (unless either the function is already zero or the current input is large). Moreover, the rate of decay of $V(x(t))$ depends on the state and on the value of $V(x(t))$ (the main role of α_3 is to allow for slower convergence if $V(x(t))$ is very small or if $x(t)$ is very large; the inequality can be restated in various alternative ways). We do not yet know if IOS implies the existence of one V (independent of the input level r) with the stated properties. For the case of no inputs, however, applying this theorem with $r = 0$ one has the following, apparently new, result for OS:

Corollary 2.4 *The following statements are equivalent for systems without inputs.*

1. *System (3) is* OS.
2. *The boundedness condition (4) holds and there exist* $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, *and* $\alpha_3 \in \mathcal{KL}$, *and a smooth function* V *so that:*
   - $\alpha_1(|h(\xi)|) \leq V(\xi) \leq \alpha_2(|\xi|)$ *for all* $\xi \in \mathbb{R}^n$*;*
   - $DV(\xi)f(\xi) \leq -\alpha_3(V(\xi), |\xi|)$ *for all* $\xi \in \mathbb{R}^n$.

For this case, no inputs, the above Lyapunov property had already appeared in the literature, but only as a *sufficient* condition; see [2].

Of course, in general, if there is a V that works for all r, then the system is IOS. Since this property is probably the most useful one for verification, and in any case because the (easy) result is needed as a step in the proof of Theorem 1, we state the sufficient condition separately:

Proposition 2.5 Assume that the system (1) is UBIBS and that there exists a smooth function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ so that the following two properties hold:

- There exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that

$$\alpha_1(|h(\xi)|) \leq V(\xi) \leq \alpha_2(|\xi|), \qquad \forall \xi \in \mathbb{R}^n. \quad (7)$$

- There exist $\chi \in \mathcal{K}$ and $\alpha_3 \in \mathcal{KL}$ such that for all $\xi \in \mathbb{R}^n$ and all $\mu \in \mathbb{R}^m$:

$$V(\xi) \geq \chi(|\mu|) \Rightarrow DV(\xi)f(\xi,\mu) \leq -\alpha_3(V(\xi), |\xi|). \quad (8)$$

Then, the system is IOS.

2.2 Robust Output Feedback

In the study of the ISS property in [12], a central role was played by the equivalence between ISS and a robust stability property. Even though the analogous equivalence does not hold for IOS, it is nonetheless of interest to study the connections with the corresponding notion. Surprisingly, the notion that obtains has an elegant characterization in Lyapunov function terms.

To each given system (1) and each smooth function λ, we associate the following system with inputs $d(\cdot)$:

$$\dot{x} = g(x,d) := f(x, d\lambda(|y|)), \quad y = h(x), \quad (9)$$

where $d \in \mathcal{M}_\Omega$ with $\Omega = [-1,1]^m$. We will be using $x_\lambda(\cdot,\xi,d)$ (and $y_\lambda(\cdot,\xi,d)$, respectively) to denote the trajectory (and the output function respectively) of (9) corresponding to each initial state ξ, each function λ, and each input $d(\cdot)$.

Definition 2.6 We say that system (1) is *robustly output stable* (ROS) if it is UBIBS and there exists a smooth $\mathcal{K}_\infty$-function λ such that the corresponding system (9) is OS uniformly (UOS) with respect to all $d \in \mathcal{M}_\Omega$, that is,

$$|y_\lambda(t,\xi,d)| \leq \beta(|\xi|, t), \qquad \forall t \geq 0, \quad (10)$$

for all $\xi \in \mathbb{R}^n$ and all $d \in \mathcal{M}_\Omega$.

Observe that, for systems (3) with no controls, the ROS property is the same as OS. In general, just one implication holds:

Lemma 2.7 If a system is IOS, then it is ROS.

We now turn to a Lyapunov-like property that is closely related to the one used in Proposition 2.5: the difference between them lies in the conditions in (8) and (12) which guarantee decrease of V.

Definition 2.8 A smooth function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is an ROS-*Lyapunov function* for system (1) if the following two properties hold:

- There exist α_1 and α_2 such that

$$\alpha_1(|h(\xi)|) \leq V(\xi) \leq \alpha_2(|\xi|), \qquad \forall \xi \in \mathbb{R}^n; \quad (11)$$

- There exist $\chi \in \mathcal{K}$ and $\alpha_3 \in \mathcal{KL}$ such that

$$|h(\xi)| \geq \chi(|\mu|) \Rightarrow DV(\xi)f(\xi,\mu) \leq -\alpha_3(V(\xi), |\xi|) \quad (12)$$

  for all $\xi \in \mathbb{R}^n$ and all $\mu \in \mathbb{R}^m$. □

Our main result regarding the ROS property is:

Theorem 2 *A system (1) is* ROS *if and only if it is* UBIBS *and it admits an* ROS*-Lyapunov function.*

3 Appendix

In this appendix, we provide sketches of the proofs of several of the results (the full paper will be found in Web page cited in the bibliography). As it is the generally in Lyapunov theory, it easier to establish sufficiency statements than converse theorems, in which V must be constructed. The following is the key technical result that underlies all proofs.

Lemma 3.1 Consider a UBIBS system (1). Let Ω be a compact subset of $\mathbb{R}^m$, and let $b \geq 0$. Then the following two properties are equivalent:

1. There exists a $\mathcal{KL}$-function β such that
$$|y(t,\xi,u)| \leq \beta(|\xi|, t) + b, \qquad \forall\, t \geq 0, \qquad (13)$$
for all $\xi \in \mathbb{R}^n$ and all $u \in \mathcal{M}_\Omega$.

2. There exist continuous functions $\omega : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that the following hold:
   - $|h(\xi)| \leq \omega(\xi) + b$ for all $\xi \in \mathbb{R}^n$;
   - V is locally Lipschitz on the set $\{\xi : V(\xi) \neq 0\}$;
   - there exist $c_1, c_2 > 0$ such that
   $$c_1\omega(\xi) \leq V(\xi) \leq c_2\omega(\xi) \qquad (14)$$
   for all $\xi \in \mathbb{R}^n$;
   - there exists $\alpha_3 \in \mathcal{KL}$ such that
   $$DV(\xi)f(\xi,\mu) \leq -\alpha_3(V(\xi), |\xi|), \qquad (15)$$
   for all $\mu \in \Omega$, for almost all ξ such that $V(\xi) \neq 0$.

The proof of the result is fairly technical and long. The complete proof will be given in the full paper. We remark here that the choice of c_1, c_2 and α_3 only depends on the decay function β, and it is independent of b. Below we show how to prove, with the help of Lemma 3.1, the results stated in Section 2.

3.1 Proof of Theorem 1

Proof of [2 $\Rightarrow$ 1]. Let α_i $(i = 1, 2, 3)$ and χ be given as in Theorem 1. Fix $r \geq 0$. Let V_r be the corresponding function satisfying all the conditions in the theorem. Pick $\xi \in \mathbb{R}^n$ and an input u with $\|u\| = r$, and use $x(t) := x(t,\xi,u)$. Consider the set
$$S_r = \{\xi \in \mathbb{R}^n : \quad V_r(\xi) \leq \chi(r)\}.$$
It then follows from condition 2 in the theorem that whenever $x(t) \notin S_r$,
$$\frac{d}{dt}V_r(x(t)) \leq -\alpha_3(V_r(x(t)), |x(t)|).$$
Following the same steps as in page 441 of [9], one can show that if $x(t_0) \in S_r$, then $x(t) \in S_r$ for all $t \geq t_0$.

Observe that since the system is UBIBS, there is some $\sigma \in \mathcal{K}$ such that (2) holds, i.e.,
$$|x(t)| \leq \max\{\sigma(|\xi|),\ \sigma(r)\} \qquad \forall\, t \geq 0.$$
We now let $t_1 = \min\{t \geq 0 : \ x(t) \in S_r\}$. If $t_1 < \infty$, then
$$V_r(x(t)) \leq \chi(r), \qquad \forall\, t \geq t_1. \qquad (16)$$
On $[0, t_1)$, it holds that
$$\dot{V}_r(x(t)) \leq -\alpha_3(V_r(x(t)), |x(t)|) \leq -\alpha_3(V_r(x(t)), c),$$
where $c = \max\{\sigma(|\xi|),\ \sigma(r)\}$. By a generalized comparison principle, one knows that there exists a family $\{\beta_r(s,t)\}_{r\geq 0}$ (which only depends on the $\mathcal{KL}$-function α_3) of $\mathcal{KL}$ functions with the property that $\beta_{r_1}(s,t) \leq \beta_{r_2}(s,t)$ for all s, t if $r_1 \leq r_2$ such that
$$V_r(x(t)) \leq \beta_c(V(\xi), t) \quad \forall\, t \in [0, t_1). \qquad (17)$$
Noticing that
$$\begin{aligned} &\beta_c(V(\xi), t) \leq \beta_c(\alpha_2(|\xi|), t) \\ \leq\ &\max\{\beta_{\sigma(|\xi|)}(\alpha_2(|\xi|), t),\ \beta_{\sigma(r)}(\alpha_2(r), t)\} \end{aligned}$$
for all $\xi \in \mathbb{R}^n$ and $r, t \geq 0$ (consider two cases: $|\xi| \geq r$ and $|\xi| < r$), one obtains
$$V_r(x(t)) \leq \max\{\bar{\beta}(|\xi|, t),\ \gamma_0(r)\} \quad \forall\, t \in [0, t_1), \quad (18)$$
where $\bar{\beta}(s,t) := \beta_{\sigma(s)}(\alpha_2(s), t)$, and $\gamma(s) := \beta_{\sigma(s)}(\alpha_2(s), 0)$. Combining (17) and (18), one obtains
$$V_r(x(t)) \leq \max\{\bar{\beta}(|\xi|, t),\ \gamma_0(\|u\|),\ \chi(\|u\|)\},\ \forall\, t \geq 0,$$
from which it follows that
$$|h(x(t))| \leq \max\{\hat{\beta}(|\xi|, t),\ \gamma(\|u\|)\}, \quad \forall\, t \geq 0, \quad (19)$$
where $\hat{\beta}(s,t) := \alpha_1^{-1}(\bar{\beta}(s,t))$, and
$$\gamma(s) = \max\{\alpha_1^{-1}(\gamma_0(s)),\ \alpha_1^{-1}(\chi(s))\}.$$

Proof of [2 $\Rightarrow$ 1]. Fix $r \geq 0$, and let $\mathcal{M}_r$ denote the set of inputs u with $\|u\| \leq r$. Then
$$|x(t,\xi,u)| \leq \beta(|\xi|, t) + \gamma(r), \qquad \forall\, t \geq 0,$$
for all ξ and all $u \in \mathcal{M}_r$. An immediate consequence of Lemma 3.1 (with $b = \gamma(r)$) is the following:

Lemma 3.2 Assume that system (1) is IOS. Then there exist $c_1, c_2 > 0$ and $\alpha_3 \in \mathcal{KL}$ such that for any $r \geq 0$, there exist continuous functions $\omega : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, $W : \mathbb{R}^n \to R_{\geq 0}$ so that the following holds:

- $|h(\xi)| \leq \omega(\xi) + \gamma(r)$ for all $\xi \in \mathbb{R}^n$;
- $c_1\omega(\xi) \leq W(\xi) \leq c_2\omega(\xi)$;
- W is locally Lipschitz on the set $\{\xi : \ W(\xi) \neq 0\}$;
- it holds that $DW(\xi)f(\xi,\mu) < -\alpha_3(W(\xi), |\xi|)$ for all $|\mu| \leq r$ and almost all ξ such that $W(\xi) \neq 0$.

We now continue to prove Theorem 1. Let c_1, c_2 and α_3 be as in above. Fix $r \geq 0$. Let ω_r and W_r be as in Lemma 3.2 corresponding to the fixed value r. Let D_r be the set defined by
$$D_r = \{\xi : \ W_r(\xi) = 0\}.$$

By [8, Theorem B.1], there exists a smooth function $\hat{W}_r(\xi)$ defined on $\mathbb{R}^n \setminus D_r$ such that

$$\left|\hat{W}_r(\xi) - W_r(\xi)\right| \le W_r(\xi)/2, \qquad \forall\, \xi \notin D_r,$$

and

$$D\hat{W}_r(\xi)f(\xi,\mu) \le -\alpha_3(W_r(\xi),|\xi|)/2, \ \text{ a.e. } \xi \notin D_r.$$

Extend $\hat{W}_r$ to $\mathbb{R}^n$ by letting $\hat{W}_r(\xi) = 0$ for all $\xi \in D_r$. Then $\hat{W}_r$ is continuous everywhere,

$$\hat{c}_1(\hat{\omega}_r(\xi) - \gamma(r)) \le \hat{W}_r(\xi) \le \hat{\alpha}_2(|\xi|), \qquad \forall\, \xi \in \mathbb{R}^n,$$

where $\hat{c}_1 = c_1/2$, $\hat{\alpha}_2(s) = 2c_2\beta(s,0)$, $\hat{\omega}_r(\xi) = \omega_r(\xi) + \gamma(r)$, and

$$D\hat{W}_r(\xi)f(\xi,\mu) \le -\hat{\alpha}_3(W_r(\xi),|\xi|), \ \text{ whenever } \hat{\omega}_r(\xi) \ne 0,$$

where $\hat{\alpha}_3(s,r) = \alpha(s/2,r)/2$.

Let, for each $r > 0$, $\varphi_r : \mathbb{R}^n \to [0,1]$ be such that

$$\varphi_r(\xi) = \begin{cases} 1, & \text{if } \hat{\omega}_r(\xi) \ge 2\gamma(r), \\ 0, & \text{if } \hat{\omega}_r(\xi) \le \frac{3\gamma(r)}{2}, \end{cases}$$

and let $\psi : \mathbb{R}^n \to \mathbb{R}_{\ge o}$ be a smooth function such that $\rho(|h(\xi)|) \le \psi(\xi) \le \hat{c}_1 |h(\xi)|/3$ for some $\mathcal{K}_\infty$-function ρ. Define

$$V_r(\xi) = \varphi_r(\xi)\hat{W}_r(\xi) + (1 - \varphi_r(\xi))\psi(\xi).$$

Then V_r is smooth everywhere,

$$V_r(\xi) \ge \begin{cases} \hat{c}_1\hat{\omega}_r(\xi)/2, & \text{if } \hat{\omega}_r(\xi) \ge 2\gamma(r), \\ \rho(|h(\xi)|), & \text{if } \hat{\omega}_r(\xi) \le 3\gamma(r)/2, \end{cases}$$

and for $3\gamma(r)/2 \le \hat{\omega}_r(\xi) \le 2\gamma(r)$, one has

$$\begin{aligned} V_r(\xi) &\ge \hat{c}_1\varphi_r(\xi)(\hat{\omega}_r(\xi) - \gamma(r)) + (1 - \varphi_r(\xi))\psi(\xi) \\ &\ge \frac{\hat{c}_1\varphi_r(\xi)\hat{\omega}_r(\xi)}{3} + \psi(\xi) - \varphi_r(\xi)\frac{\hat{c}_1\,|h(\xi)|}{3} \\ &\ge \rho(|h(\xi)|). \end{aligned}$$

Combining this with the fact that $V_r(\xi) \le \frac{\hat{c}_1|h(\xi)|}{3} + \hat{\alpha}_2(|\xi|)$, one obtains:

$$\widetilde{\alpha}_1(|h(\xi)|) \le V_r(\xi) \le \widetilde{\alpha}_2(|\xi|), \qquad \forall\, \xi \in \mathbb{R}^n,$$

where $\widetilde{\alpha}_1(s) = \min\{\rho(s),\ \hat{c}_1 s\}$, and $\widetilde{\alpha}_2(s) = \hat{\alpha}_2(s) + \rho_1(s)$, and where ρ_1 is such a $\mathcal{K}$-function such that $\frac{\hat{c}_1|h(\xi)|}{3} \le \rho_1(|\xi|)$ for all ξ. Since $V_r(\xi) = W_r(\xi)$ when $\hat{\omega}(\xi) \ge 2\gamma(r)$, it holds that

$$\hat{\omega}(\xi) \ge 2\gamma(r) \Rightarrow DV_r(\xi)f(\xi,\mu) \le -\hat{\alpha}_3(V_r(\xi),|\xi|), \tag{20}$$

for all $|\mu| \le r$. Observe that when $\hat{\omega}(\xi) \ge 2\gamma(r)$, it holds that

$$V_r(\xi) = \hat{W}_r(\xi) \le 2c_2\omega(\xi) \le 2c_2\hat{\omega}(\xi),$$

and hence, it follows from (20) that

$$V_r(\xi) \ge \gamma_1(r) \Rightarrow DV_r(\xi)f(\xi,\mu) \le -\hat{\alpha}_3(V_r(\xi),|\xi|)$$

for all $|\mu| \le r$, where $\gamma_1(s) = \frac{\gamma(s)}{c_2}$. This finishes the construction of smooth V_r's for $r > 0$. To get a smooth function V_0 from $\hat{W}_0$, one can follow exactly the same steps as in the smoothing arguments used in [8]. ∎

3.2 Proof of Theorem 2

Proof of the sufficiency. Let V be an ROS-Lyapunov function for system (1) with α_i ($i = 1,2,3$) and χ as in (11) and (12). Without loss of generality, we may assume that $\chi \in \mathcal{K}_\infty$. With $\lambda = \chi^{-1}$, one can rewrite (12) as

$$DV(\xi)f(\xi,\mu\lambda(|h(\xi)|)) \le -\alpha_3(V(\xi),|\xi|)$$

for all $\xi \in \mathbb{R}^n$, all $|\mu| \le 1$. This implies that for any ξ and any $d \in \mathcal{M}_\Omega$ with $\Omega = [-1,\,1]^m$, the corresponding trajectory $x_\lambda(t) := x_\lambda(t,\xi,d)$ of system (9) satisfies the following:

$$\frac{d}{dt}V(x_\lambda(t)) \le -\alpha_3(V(x_\lambda(t)),\,|x_\lambda(t)|), \qquad \forall\, t \ge 0. \tag{21}$$

It follows immediately that $V(x_\lambda(t)) \le V(\xi)$ for all $t \ge 0$, which, in turn, implies that

$$|h(x_\lambda(t))| \le \sigma_4(|\xi|),\ \forall\, t \ge 0, \tag{22}$$

where $\sigma_4 = \alpha_1^{-1} \circ \alpha_2$. Since system (1) is UBIBS, there is some $\sigma \in \mathcal{K}$ such that

$$|x_\lambda(t)| \le \max\{\sigma(|\xi|), \sigma(\|u_d\|)\},$$

where $u_d(t) = d(t)\lambda(|y_\lambda(t)|)$. Combining this with (22), one sees $|x_\lambda(t)| \le \hat{\sigma}(|\xi|)$ for all $t \ge 0$, where $\hat{\sigma}(s) = \max\{\sigma(s),\ \lambda(\sigma_4(s))\}$. Putting this back into (21), we have

$$\frac{d}{dt}V(x_\lambda(t)) \le -\alpha_3(V(x_\lambda(t)),\,\hat{\sigma}(|\xi|)), \qquad \forall\, t \ge 0.$$

Again, by a generalized comparison principle, one knows that there is some $\beta \in \mathcal{KL}$ depending only on α_3 and $\hat{\sigma}$, such that $V(x_\lambda(t)) \le \beta(|\xi|,t)$ for all $t \ge 0$, from which it follows that

$$|h(x_\lambda(t))| \le \hat{\beta}(|\xi|,t), \qquad \forall\, t \ge 0,$$

where $\hat{\beta}(s,t) = \alpha_1^{-1}[\beta(\alpha_2(s),t)]$.

The necessity part of Theorem 2 is an easy consequence of Lemma 3.1. Due to space limitations, we must omit the detailed arguments here.

3.3 Proof of Lemma 2.7

Assume that system (1) is UBIBS with estimation (2). Without loss of generality, we may assume that $\sigma(s) \ge s$ for all $s \ge 0$. Let κ_0 any $\mathcal{K}_\infty$-function such that $\kappa_0(2\sigma(s)) \le s/4$.

Using the same arguments as used in [12], one can show that if $|u(t)| \le \kappa_0(|x(t,\xi,u)|)$ for almost all $t \ge 0$, then $|u(t)| \le \frac{|\xi|}{2}$ for almost all $t \ge 0$.

Let $E := \{(\xi,\mu) : |\mu| \le \kappa_0(|\xi|)\}$. The above conclusion implies that if $(x(t,\xi,u),u(t)) \in E$ for almost all $t \geq 0$, then $|x(t,\xi,d)| \leq \sigma(|\xi|)$ for all $t \geq 0$.

Assume now that system (1) is IOS with decay estimation (5). Let $t_1 > 0$. Then, for any $t \geq t_1$,

$$|y(t,\xi,d)| \leq \beta(|\xi_1|, t-t_1) + \gamma(\|u\|_{[t_1,\infty)}),$$

where $\xi_1 = x(t_1,\xi,u)$, and $\|u\|_{[t,\infty)}$ denotes the L_∞ norm of u restricted to $[t,\infty)$. Thus, if $(x(t,\xi,u),u(t)) \in E$ a.e., then the following holds:

$$|y(t,\xi,d)| \leq \beta_1(|\xi|, t-t_1) + \gamma(\|u\|_{[t_1,\infty]}), \quad \forall t \geq 0,$$

where $\beta_1(s,t) = \beta(\sigma(s),t) \in \mathcal{KL}$, and in particular,

$$|y(t,\xi,d)| \leq \beta_1(|\xi|, t/2) + \gamma(\|u\|_{[t/2,\infty)}), \quad \forall t \geq 0,$$

if $(x(t),u(t)) \in E$ a.e. Without loss of generality, we assume that $\gamma \in \mathcal{K}_\infty$. Now let

$$E_1 = \{(\xi,\mu) \in E : |\mu| \leq \gamma^{-1}(|h(\xi)|/2)\} \subset E.$$

If $(x(t,\xi,u),u(t)) \in E_1$ a.e., then

$$|y(t,\xi,d)| \leq \beta_1(|\xi|, t/2) + \frac{\|y\|_{[t/2,\infty)}}{2}, \quad \forall t \geq 0.$$

Let $\sigma_3 \in \mathcal{K}_\infty$ such that $|h(\xi)| \leq \sigma_3(|\xi|)$. Then

$$E_1 \supset \{(\xi,\mu) : |\mu| \leq \kappa(|h(\xi)|)\},$$

where $\kappa(s) = \min\{\gamma^{-1}(s/2),\ \kappa_0(\sigma_3^{-1}(s))\}$. Consider the system

$$\dot{x} = f(x, d\kappa(|y|)), \quad y = h(x), \tag{23}$$

with $d \in \mathcal{M}_\Omega$. Pick any $d \in \mathcal{M}_\Omega$ and any ξ. Let $x_\kappa(t)$ denote the corresponding trajectory of (23), and $y_\kappa(t)$ the corresponding output function. Then $(x_\kappa(t), d(t)y_\kappa(t)) \in E_1$ a.e., and hence,

$$|y_\kappa(t)| \leq \beta_2(|\xi|, t) + \frac{\|y_\kappa\|_{[t/2,\infty)}}{2}, \quad \forall t \geq 0,$$

where $\beta_2(s,t) = \beta_1(s,t/2) \in \mathcal{KL}$. By [4, Lemma A.1], there is some $\mathcal{KL}$-function $\hat{\beta}$, which only depends on β_2, such that $|y_\kappa(t)| \leq \hat{\beta}(|\xi|,t)$ for all $t \geq 0$. Hence, system (23) is OS uniformly on all $d \in \mathcal{M}_\Omega$. ∎

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