A remark on ω-limit sets for non-expansive dynamics

Alon Duvall[1] and Eduardo D. Sontag[2]

***Abstract*— In this paper, we study systems of time-invariant ordinary differential equations whose flows are non-expansive with respect to a norm, meaning that the distance between solutions may not increase. Since non-expansiveness (and contractivity) are norm-dependent notions, the topology of ω-limit sets of solutions may depend on the norm. For example, and at least for systems defined by real-analytic vector fields, the only possible ω-limit sets of systems that are non-expansive with respect to polyhedral norms (such as ℓ^p norms with $p = 1$ or $p = \infty$) are equilibria. In contrast, for non-expansive systems with respect to Euclidean (ℓ^2) norm, other limit sets may arise (such as multi-dimensional tori): for example linear harmonic oscillators are non-expansive (and even isometric) flows, yet have periodic orbits as ω-limit sets. This paper shows that the Euclidean linear case is what can be expected in general: for flows that are non-expansive with respect to any strictly convex norm (such as ℓ^p for any $p \neq 1, \infty$), and if there is at least one bounded solution, then the ω-limit set of every trajectory is also an ω-limit set of a linear time-invariant system.**

I. Introduction

Contraction theory concerns dynamical systems which posses some kind of metric, typically arising from a norm, such that for every two trajectories, their distance is nonincreasing or even decreasing over time. The use of contraction analysis in control theory was pioneered by Slotine and collaborators [1]. Expositions of contractivity in dynamical systems can be found for example in [2], [3], and [4], which also show that in general non-Euclidean norms must be considered when analyzing nonlinear dynamics. The notion of contraction has also been generalized to Finsler manifolds in [5]. Most work deals with cases when the distance between trajectories is strictly decreasing, though sometimes the situation arises where all we can say is that this distance is nonincreasing. Our paper is concerned with dynamical conclusions that one can draw when a dynamical system has a merely nonincreasing norm.

Contraction theory has many connections to control theory and dynamical systems, as well as other fields. It has applications to data-driven control [6], reaction diffusion systems [7], Hopfield neural networks [8], Riemannian manifolds [9], network systems [10], and system safety [11]. Establishing contractivity of a system allows one to conclude many desirable stability properties. This makes contraction theory a useful tool in the context of certifying robustness guarantees. Our main results stated informally are as follows. In the following suppose we are given a system that possesses at least one bounded trajectory. If a system is non-expansive with respect to some norm, then solutions will converge to a global attractor set on which the system evolves isometrically. The structure of these ω-limit sets is dependent on the particular choice of norm for which the system is non-expansive. If the norm is strictly convex then the equilibrium set is convex, and the system is equivalent to a linear system on the global attractor. In this case, we show that each ω-limit set has the structure of an n-torus for some integer n. This differs from the situation of polyhedral norms for analytic vector fields. In this case the ω-limit sets are always single points. In $\mathbb{R}^2$, weighted ℓ^2 norms are the only norms for which a non-expansive system (with respect to a weighted ℓ^2 norm) does not necessarily converge to the equilibrium set. At the end we describe some examples.

II. Background

In the following we will describe norms and dynamical systems with special properties relating to the norm. We will assume that we have an autonomous system $\dot{x} = f(x)$ where $x \in \mathbb{R}^n$ and $f(\cdot)$ is C^1. We assume that we are given a particular norm $\|.\|$ on $\mathbb{R}^n$. We will define the forward time evolution of the system $\dot{x} = f(x)$ to be ϕ_t. We assume that ϕ_t is defined for all $t \geq 0$. Given a vector field $f(x)$, we let the Jacobian evaluated at a point x be $\mathcal{J}_f(x)$.

Now we will recall a few basic definitions that will be used in the sequel. A *state space* X is a forward invariant set for the system. An ω *limit set* of a point x is the set of points $\cap_{t\geq 0}\overline{\cup_{s\geq t}\phi_s(x)}$. A system $\dot{x} = f(x)$ is *non-expansive* with respect to a norm $\|.\|$ if for all $t > 0$ and all x, y in the system's state space we have that $\|\phi_t(x) - \phi_t(y)\| \leq \|x - y\|$. In other words, the flow maps are Lipschitz with Lipschitz constant 1. The *global attractor* of a system $\dot{x} = f(x)$ (relative to the state space X) is the set $\mathbb{A} = \cap_{t\geq 0}\phi_t(X)$. An *isometry* of a normed vector space V is a mapping $F : V \to V$ such that $\|x - y\| = \|F(x) - F(y)\|$ for all $x, y \in V$. By GL_n we mean the real general linear group of degree n. A *discrete subgroup* of GL_n is a group $G \subseteq GL_n$ such that for every $g \in G$ there exists an open ball O_g such that $O_g \cap G = g$. For any positive integer n we indicate the *n-torus* by the n product $S^1 \times S^1 \times ... \times S^1 = (S^1)^n$ where S^1 is the circle.

III. Some results on nonexpansivity

While our results apply to non-compact state spaces, we can motivate working on compact state spaces by using Corollary 1 below, which says that, under minimal assumptions, we can restrict analysis to a sufficiently large compact ball which contains the initial conditions of interest. The result

This work was partially supported by grants AFOSR FA9550-21-1-0289 and NSF/DMS-2052455

[1]Northeastern University duvall.a@northeastern.edu

[2]Northeastern University e.sontag@northeastern.edu, sontag@sontaglab.org

will follow from Proposition 1. First we recall a well-known lemma that extends Brouwer's fixed point theorem to flows (Yorke's fixed point theorem in [4]); for completeness, we provide a self-contained proof.

Lemma 1. *Suppose we have a C^1 vector field $f(x)$ and a compact and convex forward invariant set X. Then $f(x)$ has an equilibrium on X.*

Proof. Suppose $f(x)$ did not have an equilibrium on X. Then for any point p there exists a time t_p such that for $t < t_p$ that $\phi_t(p) \neq p$, i.e., p is taken to a different point. In fact, this t_p also works for all points in a neighborhood of p (this can be seen via the Flow-box Theorem). Cover X with all such neighborhoods. Since X is compact we can then pick finitely many of these neighborhoods to cover x. Then we can take t_f to be the minimum of all the times corresponding to these neighborhoods. Thus for $t < t_f$ we have that ϕ_t has no fixed points, contradicting Brouwer's fixed point theorem. Thus f must have a fixed point. □

The following result, and the main ideas of its proof, are given as Theorem 19 in [10]. We provide a streamlined proof for completeness.

Proposition 1. *Suppose we have a non-expansive time-invariant system $\dot{x} = f(x)$ which has a convex forward invariant set. Then exactly one of the following two conditions holds:*

1) *Every trajectory of the system is unbounded.*
2) *The system has an equilibrium point x^* (that is $f(x^*) = 0$), and every trajectory is bounded.*

Proof. First assume we have a bounded trajectory with initial point x. Thus we can consider $\omega(x)$, the ω-limit set of x, which is a nonempty backward and forward invariant compact set for the system. For arbitrary $\epsilon > 0$ define $B_\epsilon(p) = \{y \in \mathbb{R}^n | \|y - p\| \leq \epsilon\}$. Consider $C_\epsilon = \cap_{p \in \omega(x)} B_\epsilon(p)$. Note that C_ϵ is convex and compact, since it is the intersection of convex and compact sets. If $y \in C_\epsilon$ then we must have for all $p \in \omega(x)$ that $\|y - p\| \leq \epsilon$, due to the definition of C_ϵ. Fix an arbitrary $t \geq 0$ and $p \in \omega(x)$. Since $\omega(x)$ is backward invariant there exists $p' = \phi_{-t}(p) \in \omega(x)$ such that $\phi_t(p') = p$. Since the system is non-expansive we must have that $\|\phi_t(y) - \phi_t(p')\| \leq \|y - p'\| \leq \epsilon$. From this we have that

$$\|\phi_t(y) - p\| = \|\phi_t(y) - \phi_t(p')\| \leq \epsilon.$$

Since p was arbitrary, we must have that $\phi_t(y) \in C_\epsilon$ for all $t \geq 0$, and so C_ϵ is forward invariant for all $\epsilon \geq 0$ (if it is empty the statement is trivial).
Now we can pick ϵ large enough such that C_ϵ is nonempty (which is clearly possible since $\omega(x)$ is compact). Then we can apply Lemma 1 to conclude that $\dot{x} = f(x)$ has a fixed point in C_ϵ. □

Note that a non-expansive system can have unbounded trajectories, such as the system $\dot{x} = 1$. This is non-expansive with respect to any norm.

The following corollary follows immediately from Proposition 1.

Corollary 1. *Suppose we have a system that is non-expansive with respect to a norm $\|.\|$ and that has a precompact trajectory. Then the system has at least one equilibrium point p, and every norm ball $B_{p,d} = \{x \in \mathbb{R}^n | \|p - x\| \leq d\}$ is a compact forward invariant set.*

Thus, for the remainder of this section, we will make the assumption, restricting if necessary to balls around an equilibrium, that *all state spaces X we consider are convex and compact.*

A. Compact state space

Lemma 2. *Suppose we have a non-expansive dynamical system $\dot{x} = f(x)$ with a C^1 vector field $f(x)$ and a compact forward invariant state space X. Then for any $\epsilon > 0$ there exists $T > 0$ such that for all $t, s > T$ and for all $x, y \in X$ we have that $|\|\phi_t(x) - \phi_t(y)\| - \|\phi_s(x) - \phi_s(y)\|| < \epsilon$.*

Proof. For the following let $n \in \mathbb{Z}$. Consider the sequence of functions $d_n : X \times X \to \mathbb{R}_{\geq 0}$ for $n \geq 0$ (here we give $X \times X$ the sup product metric) defined by

$$d_n(x, y) = \|\phi_n(x) - \phi_n(y)\|.$$

We have that $X \times X$ is a compact set. Note that d_n satisfies the triangle inequality (since the norm satisfies it) and is symmetric. Due to the nonexpansivity of ϕ_n we have that $d_n(x, y)$ is monotonically decreasing in n for any given (x, y). Due to the nonnegativity of norms we also have that $d_n \geq 0$ for each n and so d_n is bounded below. Thus as $n \to \infty$ we have pointwise convergence to some function d. Note that d_n is also a continuous function for each n. This is due to ϕ_n and the norm function both being continuous.
Lastly, we note that $d(x, y)$ is a continuous function on $X \times X$. Indeed, by the triangle inequality we have that for all $x, y, x', y' \in X$:

$$d_n(x', y') - d_n(x', x) - d_n(y', y) \leq d_n(x, y)$$
$$d_n(x, y) \leq d_n(x', x) + d_n(x', y') + d_n(y', y).$$

Suppose that $(x, y), (x', y')$ are close to each other in $X \times X$, i.e., $\max\{\|x - x'\|, \|y - y'\|\} < \epsilon$. Due to nonexpansivity, we have that $\max\{\|\phi_n(x) - \phi_n(x')\|, \|\phi_n(y) - \phi_n(y')\|\} < \epsilon$ for all $n \geq 0$. Thus we have that $0 \leq d_n(x, x') < \epsilon$ and $0 \leq d_n(y, y') < \epsilon$ for all $n \geq 0$. Using these bounds in the previous inequality, we now have that

$$d_n(x', y') - 2\epsilon < d_n(x, y) < d_n(x', y') + 2\epsilon.$$

Thus we have that $|d_n(x, y) - d_n(x', y')| < 2\epsilon$ for all $n \geq 0$. Taking the limit in n, we see that $|d(x, y) - d(x', y')| \leq 2\epsilon$. Thus the function d is continuous.
Now we can apply Dini's theorem and so we have that d_n in fact converges uniformly to d. Thus there exists N such that for $n \geq N$ we have that $d_n(x, y) - d(x, y) < \epsilon$ for all $x, y \in X$. Thus we also have that $d_n(x, y) - d_{n+k}(x, y) < \epsilon$ for all $n \geq N$ and all integers $k \geq 0$ (i.e., this sequence is a Cauchy sequence at each point (x, y)). □

Note this lemma says that points in the state space uniformly approach their minimum distance from each other. We then have the following:

Corollary 2. *Suppose we have a C^1 system $\dot{x} = f(x)$ with compact forward invariant state space X. Then for any real number $t \geq 0$ the time evolution operator ϕ_t is an isometry on the set $\mathbb{A} = \cap_{t \geq 0} \phi_t(X)$ (i.e., the global attractor of the system).*

Proof. Defining $d_n(x, y)$ and $d(x, y)$ as in Lemma 2, we know that for any $\epsilon > 0$ there is an integer $N > 0$ so that $d_n(x', y') - d_{n+k}(x', y') < \epsilon$ for all $x', y' \in \mathbb{A}$ and all $n > N$ and $k > 0$. Pick now any $x, y \in \mathbb{A}$ and any two integers $n > 0$ and $k > 0$ such that $n > N$. Since $\mathbb{A} \subseteq \phi_n(X)$, we have that there exist x', y' such that $\phi_n(x') = x$ and $\phi_n(y') = y$. We have that

$$\begin{aligned} &\|x - y\| - \|\phi_k(x) - \phi_k(y)\| \\ &= \|\phi_n(x') - \phi_n(y')\| - \|\phi_{k+n}(x') - \phi_{k+n}(y')\| \\ &= d_n(x', y') - d_{n+k}(x', y') < \epsilon. \end{aligned}$$

Since ϵ can be arbitrarily small, we have that $\|x - y\| - \|\phi_k(x) - \phi_k(y)\| = 0$, or $\|\phi_k(x) - \phi_k(y)\| = \|x - y\|$. Since k was arbitrary, this holds for all integers $k \geq 0$. Note that this implies, for example, for each $0 \leq t \leq 1$ that (by nonexpansivity)

$$\|\phi_0(x) - \phi_0(y)\| \geq \|\phi_t(x) - \phi_t(y)\| \geq \|\phi_1(x) - \phi_1(y)\|.$$

Since the left and right terms are equal, all the inequalities are in fact equalities. The same argument can be applied to any positive real number t.
Thus for $x, y \in \mathbb{A}$ and any real number $t \geq 0$ we have that $\|\phi_t(x) - \phi_t(y)\| = \|x - y\|$, and so the time evolution operator is an isometry on this set. □

Observe that $\mathbb{A}$ is nonempty, since it is an intersection of a decreasing family of compact sets. A key property is that every trajectory converges to $\mathbb{A}$, as shown next.

Lemma 3. *Every ω-limit set is contained in $\mathbb{A}$.*

Proof. Take an arbitrary $x \in X$. Since the state space X is compact, the solution starting from x has a nonempty compact, connected, and backward and forward invariant ω-limit set $\omega(x)$, and the solution converges to it. Pick any $y \in \omega(x)$. Then for all $t > 0$ we have that $\phi_{-t}(y) \in \omega(x) \subseteq X$ and so $y \in \phi_t(X)$. Thus $y \in \cap_{t>0} \phi_t(X) = \mathbb{A}$. Since y was arbitary, this shows that $\omega(x) \subseteq \mathbb{A}$. □

One could also derive Corollary 2 by appealing to a result from Freudenthal and Hurewicz [12] which showed that every non-expansive map from a totally bounded metric space (for example, any compact space) onto itself must be an isometry; see also [13].

B. *Strictly convex norms*

Recall that a norm $\|.\|$ is *strictly convex* if and only if whenever x and y are two distinct points with $\|x\| = r$ and $\|y\| = r$ for some $r > 0$, we have that for $0 < \alpha < 1$ then $\|\alpha x + (1 - \alpha) y\| < r$. For the case where a given norm is strictly convex we have the following uniqueness lemma:

Lemma 4. *Suppose we have a strictly convex norm $\|.\|$. Pick two distinct points x, y and any number $0 \leq a < \|x - y\|$. Then the point z that satisfies $\|x - z\| = a < \|x - y\|$ and $\|x - y\| = \|x - z\| + \|z - y\|$ exists and is unique.*

Proof. Note there exists such a point, since we can simply take $z = (1 - \frac{a}{\|x-y\|}) x + \frac{a}{\|x-y\|} y$.
If there were two points z and z' with the claimed property, consider $x - z$ and $x - z'$. Pick any number α such that $0 < \alpha < 1$. Let $z'' = \alpha z + (1 - \alpha) z'$. Note we have that

$$x - z'' = \alpha(x - z) + (1 - \alpha)(x - z')$$

and

$$y - z'' = \alpha(y - z) + (1 - \alpha)(y - z')\,.$$

By the triangle inequality we have that

$$\|x - z''\| + \|z'' - y\| \geq \|x - y\|.$$

Note that $\|x - z\| = \|x - z'\| = a$ and $\|y - z\| = \|y - z'\| = \|x - y\| - a$. By strict convexity we have that

$$\|x - z''\| = \|\alpha(x - z) + (1 - \alpha)(x - z')\| < a$$

and

$$\|y - z''\| = \|\alpha(y - z) + (1 - \alpha)(y - z')\| < \|x - y\| - a\,.$$

This gives us $\|x - z''\| + \|z'' - y\| < \|x - y\|$, contradicting the triangle inequality; thus the point is unique. □

Notice that Lemma 4 need not hold for non-strictly convex norms. For example, consider the ℓ^1 norm and $x = (0, 0)$, $y = (1, 1)$. Then with $a = 1/2$ we can pick $z_1 = (0, 1)$ and $z_2 = (1, 0)$ to satisfy the property that $\|x - y\| = 2 = 1 + 1 = \|x - z\| + \|z - y\|$.
From now on in this section, we assume that the norm being considered is strictly convex.

Lemma 5. *For $x, y \in \mathbb{A}$, $t \geq 0$ and $1 \geq \lambda \geq 0$ we have that $\phi_t(\lambda x + (1 - \lambda) y) = \lambda \phi_t(x) + (1 - \lambda) \phi_t(y)$*

Proof. Let $d(x, y) = \|x - y\|$. Let $z = \lambda x + (1 - \lambda) y$, $d(z, x) = a$ and $d(z, y) = b$. We have that $d(x, y) = d(z, y) + d(z, x) = a + b$. Note that z is the unique point (due to Lemma 4) such that $d(z, x)$ and $d(z, y)$ take on these real values a and b, respectively.
By Corollary 2, we have that $d(\phi_t(x), \phi_t(y)) = d(x, y) = a + b$. Since z might not be in $\mathbb{A}$, we cannot yet assert the isometric relationships $d(\phi_t(z), \phi_t(x)) = a$ or $d(\phi_t(z), \phi_t(y)) = b$. However, by nonexpansivity we have that $d(\phi_t(z), \phi_t(x)) \leq d(z, x) = a$, and $d(\phi_t(z), \phi_t(y)) \leq d(z, y) = b$. By the triangle inequality we have that

$$\begin{aligned} a + b &= d(\phi_t(x), \phi_t(y))) \\ &\leq d(\phi_t(z), \phi_t(x)) + d(\phi_t(z), \phi_t(y)) \\ &\leq a + b. \end{aligned}$$

Since the left and right hand are the same we must have that $d(\phi_t(z), \phi_t(x)) = a$ and $d(\phi_t(z), \phi_t(y)) = b$, as desired.

Thus $\phi_t(z)$ satisfies the conditions in Lemma 4 where x and y are replaced with $\phi_t(x)$ and $\phi_t(y)$, respectively. This implies $\phi_t(z) = \lambda\phi_t(x) + (1-\lambda)\phi_t(y)$. Indeed, we have that $d(\phi_t(z), \phi_t(x)) = \|\phi_t(z) - \phi_t(x)\| = (1-\lambda)\|\phi_t(y) - \phi_t(x)\| = (1-\lambda)\|y-x\| = a$ and similarly we have that $d(\phi_t(z), \phi_t(y)) = b$. □

Lemma 6. *The set $\mathbb{A}$ is backward and forward invariant.*

Proof. First observe that, for any $s > t$, $\phi_s(X) \subseteq \phi_t(X)$. Indeed, if $x \in \phi_s(X)$ then $x = \phi_s(z) = \phi_t(\phi_{s-t}(z))$ for some $z \in X$. Thus $x = \phi_t(y)$, with $y := \phi_{s-t}(z)$.
Now recall $\mathbb{A} = \cap_{t>0}\phi_t(X)$. By our previous observation for any $s > 0$ we also have that $\mathbb{A} = \cap_{t>s}\phi_t(X)$. We have that $x \in \mathbb{A}$ iff $x \in \phi_t(X)$ for all $t > 0$ iff for any $s > 0$ we have that $\phi_s(x) \in \phi_t(X)$ for $t > s$ iff $\phi_s(x) \in \cap_{t>s}\phi_t(X) = \mathbb{A}$. Thus $\mathbb{A}$ is forward invariant.
Take again $x \in \mathbb{A}$. Now for each $s > 0$ we have that $x \in \mathbb{A}$ iff $x \in \cap_{t>s}\phi_t(X)$ iff $\phi_{-s}(x) \in \cap_{t>0}\phi_t(X) = \mathbb{A}$. Since s was arbitrary, $\mathbb{A}$ must be backwards invariant as well. □

Lemma 7. *The set $\mathbb{A}$ is convex.*

Proof. Take arbitrary $x, y \in \mathbb{A}$. Pick any $0 < \lambda < 1$. We need to show that $z := \lambda x + (1-\lambda)y \in \mathbb{A}$ (z exists due to the assumed convexity of our state space). Since $\mathbb{A}$ is backwards invariant by Lemma 6, for all $t > 0$ there exist $x', y' \in \mathbb{A}$ such that $\phi_t(x') = x$ and $\phi_t(y') = y$. By Lemma 5 this means that, for each $0 < \lambda < 1$:

$$\phi_t(\lambda x' + (1-\lambda)y') = \lambda\phi_t(x') + (1-\lambda)\phi_t(y') = \lambda x + (1-\lambda)y.$$

The above equation implies that for all $t > 0$, we can find a $z' = \lambda x' + (1-\lambda)y'$ such that $\phi_t(z') = z$ (note that neither z nor z' are required to be in $\mathbb{A}$). Thus for all $t > 0$ we must have that $z \in \phi_t(\mathbb{A}) \subseteq \phi_t(X)$. Thus $z \in \cap_{t>0}\phi_t(X) = \mathbb{A}$. In other words, for all $x, y \in \mathbb{A}$ and for all $0 < \lambda < 1$ we have that $\lambda x + (1-\lambda)y \in \mathbb{A}$, as claimed. □

Remark: The assumption that our norm is strictly convex is necessary for the conclusion that $\mathbb{A}$ is convex. To see this, we show an example of a non-expansive system with respect ℓ^∞ norm but for which $\mathbb{A}$ is not convex. The system is:

$$\begin{aligned} \dot{x} &= -x + f(y) \\ \dot{y} &= 0 \end{aligned}$$

where $f : \mathbb{R} \to \mathbb{R}$ has Lipschitz constant 1, i.e. $|f(c) - f(d)| \leq |c - d|$ for all $c, d \in \mathbb{R}$. We claim that this system is non-expansive with respect to the ℓ^∞ norm. To prove this, consider two solutions $\xi(t) = (x_1(t), y_1(t))$ and $\eta(t) = (x_2(t), y_2(t))$, and write $\delta(t) := x_1(t) - x_2(t)$. Let

$$\begin{aligned} a &:= \delta(0) = x_1(0) - x_2(0) \\ b &:= y_1(0) - y_2(0) \\ b' &:= f(y_1(0)) - f(y_2(0)). \end{aligned}$$

Observe that $\dot{\delta} = -\delta + b'$, because $y_i(t) \equiv y_i(0)$. Therefore

$$\delta(t) = e^{-t}a + (1 - e^{-t})b'.$$

Applying the triangle inequality and using that $|b'| \leq |b|$ (because f has Lipschitz constant 1), it follows that

$$|\delta(t)| \leq e^{-t}|a| + (1 - e^{-t})|b| \ \ \forall t.$$

Therefore (using that $y_1(t) - y_2(t) \equiv b$):

$$\begin{aligned} \|\xi(t) - \eta(t)\|_\infty &= \max\{|\delta(t)|, |b|\} \\ &\leq \max\left\{e^{-t}|a| + (1-e^{-t})|b|, |b|\right\} \\ &\leq \max\left\{e^{-t}K + (1-e^{-t})K, |b|\right\} = K \end{aligned}$$

where $K := \max\{|a|, |b|\} = \|\xi(0) - \eta(0)\|_\infty$, which proves that the dynamics is non-expansive. Observe that all solutions converge to equilibria, so $\mathbb{A}$ is the set of equilibria, If we pick for example $f(y) := e^{-y^2}$, this equilibrium set is the non-convex set $x = e^{-y^2}$.
Thus our example is non-expansive but $\mathbb{A}$ is not convex. One may ask why the theorem does not apply. Perhaps the system is non-expansive also with respect to some strictly convex norm? One can show that this is false, but for simplicity let's restrict to any ℓ^p norm with $1 < p < \infty$, and again use $f(y) = e^{-y^2}$. Consider the equilibrium $m_1 = (1, 0)$ and the point $m_2 = (1, 1)$. Note that the line $y = 1$ is tangent to an ℓ^p norm ball centered at m_1 of radius 1. Let $B = \{x \| \|x - m_1\| = 1\}$. The line $y = 1$ can only intersect our set B at m_2 due to the norm being strictly convex. Indeed, for a point $(x_1, 1)$ we have that $\|(x_1, 1) - (1, 0)\| = (|x_1 - 1|^p + 1)^{1/p} \geq 1$ where we have equality iff $x_1 = 1$. Thus for $t > 0$ we have that $\phi_t(m_2)$ cannot be contained in B or its convex hull, and thus $\|\phi_t(m_2) - \phi_t(m_1)\| > \|m_2 - m_1\|$. Thus the system is not non-expansive. □

Since $\mathbb{A}$ is compact and convex, the vector field f restricted to $\mathbb{A}$ has an equilibrium, by Lemma 1. Without loss of generality, we can view this fixed point as the origin in $\mathbb{R}^n$, so from now on we assume that $\mathbb{A}$ contains 0 and that $f(0) = 0$. Thus also $\phi_t(0) = 0$ for all t.
In the next result, we use Mankiewicz's Theorem (see for example [14]). This theorem applies to any isometry $g : E \to Y$, where E is a nonempty subset of a real normed space X, and Y is a real normed space. If either both E and $g(E)$ are convex bodies (compact and convex with nonempty interior) or if E is open and connected and $g(E)$ is open, then g can be uniquely extended to an affine isometry $F : X \to Y$.

Lemma 8. *Let V be the linear span of $\mathbb{A}$. There exists a one-parameter family of affine isometries F_t on V such that F_t is an extension of ϕ_t restricted to $\mathbb{A}$.*

Proof. Fix any $t > 0$. We know by Lemma 2 that ϕ_t is an isometry on the convex set $\mathbb{A}$. If $\mathbb{A} = \{0\}$ then the result is trivial, so assume $\mathbb{A} \neq \{0\}$. Let $\{v_1, \ldots, v_m\}$ be a maximal linearly independent set of vectors in $\mathbb{A}$. Thus V is the span of $\{v_1, \ldots, v_m\}$. Every linear combination $p = \sum_{i=1}^m \lambda_i v_i$ with all $\lambda_i > 0$ and $\sum_{i=1}^m \lambda_i < 1$ belongs to $\mathbb{A}$ (since $p = (1 - \sum_{i=1}^m \lambda_i)0 + \sum_{i=1}^m \lambda_i v_i$ is in $\mathbb{A}$, by convexity and because $0 \in \mathbb{A}$). So $\mathbb{A}$ has a nonempty interior in V. It follows that $\mathbb{A}$ is a convex body relative to V. We now apply Mankiewicz's Theorem with $g = \phi_t$, $E = \mathbb{A}$, and $X = Y = V$. Note that $g(\mathbb{A}) = \mathbb{A}$ because

$\mathbb{A}$ is backwards complete, so that $g(\mathbb{A})$ is a convex body as needed for the theorem. Thus we have an extension to an affine transformation F_t on V. $\square$

As every ϕ_t vanishes at zero (recall that we assumed this without loss of generality), so do the mappings F_t from Lemma 8. Therefore each F_t is a linear map. Since each F_t is an isometry, F_t is nonsingular, that is, $F_t \in GL_m(\mathbb{R})$.

Lemma 9. *The mappings F_t vary continuously with t.*

Proof. Since f is a C^1 vector field, the ϕ_t mappings vary continuously with t on the compact and convex set $\mathbb{A}$. Suppose that V (i.e., the span of $\mathbb{A}$) is m dimensional. We can find m linearly independent vectors $x_1, x_2, ..., x_m$ in $\mathbb{A}$ that span V. Since $\mathbb{A}$ is forward and backwards invariant, for each $1 \leq i \leq m$ and each t, $\phi_t(x_i) \in \mathbb{A}$, and hence $F_t(x_i) = \phi_t(x_i)$. Thus $F_t(x_i)$ varies continuously with t since $\phi_t(x_i)$ varies continuously with t. We conclude that the mapping $t \rightarrow F_t$ is continuous as a map $\mathbb{R} \rightarrow GL_m(\mathbb{R})$. $\square$

Lemma 10. *We have that $F_t = e^{Bt}$ for some linear transformation B on V.*

Proof. Since $F_0 = I$ (here I is the identity transformation), $F_t F_s = F_{s+t}$, and F_t varies continuously in t , the set of transformations F_t is a one parameter subgroup of $GL_m(\mathbb{R})$. By Theorem 2.14 in [15] we can conclude that there exists a unique linear map $B \subseteq GL_m(\mathbb{C})$ such that $F_t = e^{Bt}$. Note that since $B = \frac{d}{dt} F_t|_{t=0}$ we in fact must have $B \subseteq GL_m(\mathbb{R})$ $\square$

The following is a standard property of center manifolds of linear time-invariant systems (see for example Problem 5 in Problem Set 9 in [16]); we provide a proof for completeness.

Lemma 11. *Suppose a linear system $\dot{x} = Bx$ satisfies that its trajectories are bounded and do not converge to 0. Then the matrix B has only eigenvalues with 0 real part, and it is diagonalizable.*

Proof. Note that if any eigenvalue of B had negative real part, then we can find a trajectory of the system $\dot{x} = Bx$ which converges to 0. If any had positive real part, we could find a trajectory diverging to infinity.
Note that there exists $P \in GL_m(\mathbb{C})$ such that $B = PNP^{-1}$ where N is in Jordan normal form. Note that e^{Nt} has diagonal blocks with t's on the off diagonal if any of the blocks are not diagonal matrices. This would imply again diverging trajectories, thus all the blocks must be diagonal and so B is diagonalizable. $\square$

We will call such linear differential equations *conserved* linear equations. A quadratic Lyapunov function for such systems can be constructed as usual through the solution of a Lyapunov equation (see e.g. [17]). Again for completeness, we provide a proof.

Lemma 12. *Every conserved linear system has a quadratic form P such that $\frac{d}{dt}(x^\top Px) = 0$.*

Proof. Consider a conserved linear system $\dot{x} = Bx$. Note by Lemma 11 we can diagnoalize B with a real matrix L. In other words, $L^{-1}BL$ is such that it is a skew symmetric matrix consisting of diagonal blocks of the form

$$\begin{bmatrix} 0 & \alpha \\ -\alpha & 0 \end{bmatrix}.$$

Let $P = L^\top L$. We have that

$$\begin{aligned} B^\top L^\top L + L^\top LB &= (L^{-1}BL)^\top L^\top L + L^\top L(L^{-1}BL) \\ &= L^\top B^\top L + L^\top BL \\ &= -L^\top BL + L^\top BL = 0. \end{aligned}$$

Thus $\frac{d}{dt}(x^\top Px) = x^\top(B^\top P + PB)x = 0.$ $\square$

Lemma 13. *For a conserved linear system $\dot{x} = Bx$, every point x_0 is in its own ω-limit set.*

Proof. Assume upon a linear transformation that B is block-diagonal with blocks that are either 2 by 2 skew symmetric matrices or 1 by 1 zero matrices. The trajectory of x_0 is thus $e^{Bt}x_0$ where e^{Bt} consists of 2 by 2 blocks of rotation matrices on its diagonal of the form.

$$\begin{bmatrix} \cos(\alpha_i t) & -\sin(\alpha_i t) \\ \sin(\alpha_i t) & \cos(\alpha_i t) \end{bmatrix}$$

as well as 1's in diagonal entries corresponding to zero entries in B. Put the α_i terms into a row vector $[\alpha_1, \alpha_2, ..., \alpha_l]t$ and consider this vector modulo 2π. Divide up the region $[0, 2\pi]^n$ into boxes of side length at most ϵ. Note that for any $\epsilon > 0$ and any $\delta > 0$ by the pigeonhole principle we can always find t_1 and t_2 such that $|t_1 - t_2|$ is bounded below by $\delta > 0$ and that

$$|[\alpha_1, \alpha_2, ..., \alpha_l]t_1 - [\alpha_1, \alpha_2, ..., \alpha_l]t_2| < [\epsilon, \epsilon, ..., \epsilon].$$

(Here the absolute value and comparison are done element-wise.) Indeed, the set of points $\{(t_1 + \delta j)[\alpha_1, \alpha_2, ..., \alpha_l] | j \in \mathbb{N}\}$ (taken modulo 2π) is an infinite set of points in $[0, 2\pi]^n$ and thus we can find 2 different points in the same box (from the boxes we have previously divided our region into). These two points precisely satisfy out inequality.
Thus if $t_2 > t_1$ then at time $t = t_2 - t_1$ we have that e^{Bt} is close to the identity matrix. This is due to the fact that if all the $\alpha_i t$ are close to multiples of 2π, all the 2 by 2 rotation matrices will be close to being identity matrices. Picking $\delta_i = i$ and $\epsilon_i = \frac{1}{i}$ we can always find a corresponding $t_i > \delta_i$ such that $e^{Bt_i} \rightarrow I$ as $t_i \rightarrow \infty$. In particular, for each x_0, $e^{Bt_i}x_0 \rightarrow x_0$ as $t_i \rightarrow \infty$. Thus x_0 is in its own ω-limit set. $\square$

Lemma 14. *For a conserved linear system $\dot{x} = Bx$ the trajectories are homeomorphic to an k-torus $(S^1)^k$ for some integer k.*

Proof. Via a linear transformation, we can assume that B consists of 2 by 2 blocks of skew symmetric matrices on its diagonal, and 0's elsewhere. Our trajectories are always of the form $\{e^{Bt}x_0 | t \geq 0\}$ for some initial point x_0. Whenever we have two entries of x_0 equal to 0, and they both correspond to the same block, remove this block from e^{Bt} (otherwise, these entries would simply remain 0 for the entire trajectory). Going forward we consider this reduced form of e^{Bt}.

Let T be the set of matrices which consist of 2 by 2 rotation matrices on the diagonal, 1's elsewhere on the diagonal, and 0's off the diagonal (i.e., the same general structure as e^{Bt}), seen as a Lie subgroup of an appropriate $GL(k,\mathbb{R})$. It is easy to see that T is a compact, connected, and commutative Lie group, and thus it is a torus (Theorem 11.2 in [15]). Note that the closure of $G = \{e^{Bt}|t \in \mathbb{R}\}$, call it $\bar{G}$, is a subgroup (the closure of a subgroup is still a subgroup) of T. Since $\bar{G}$ is a closed subgroup of T, it must be compact and commutative. It is also a Lie subgroup of T by the Closed Subgroup Theorem (see Theorem 20.12 in [18]). Since G is connected so is $\bar{G}$. Thus $\bar{G}$ is a compact, connected and commutative Lie subgroup of T and therefore it must be a torus itself.

Define $L = \{e^{Bt}x_0|t \in \mathbb{R}\}$. By Lemma 13 we have that x_0 is in $\omega(x_0)$ and since ω-limit sets are backward and forward invariant we must also have that $L \subseteq \omega(x_0)$ and thus since ω-limit sets are closed sets that $\bar{L} \subseteq \omega(x_0)$. Thus since $\omega(x_0) \subseteq \bar{L}$ we have that our ω-limit set is in fact precisely $\bar{L}$.

Thus we can think of $\bar{G}$ as acting on x_0, and since the stabilizer is trivial we have $\bar{L}$ is diffeomorphic to $\bar{G}$ (see Theorem 21.18 in [18]). Thus the ω-limit set of $e^{Bt}x_0$ must also be a torus. ☐

We are now ready to prove our main result:

Theorem 1. *Suppose that the dynamical system $\dot{x} = f(x)$ is non-expansive for a strictly convex norm $\|.\|$ and has at least one bounded trajectory. Then all the trajectories are bounded, and their ω-limit sets are that of some fixed conserved linear system $\dot{x} = Bx$. In particular, the ω-limit sets are homeomoprhic to $(S^1)^k$ for some integer k.*

Proof. By Lemma 8 and Lemma 10 we have that on the set $\mathbb{A}$ our dynamics must be equivalent (up to translation) to that of a linear system. Since the trajectories are bounded and do not converge to 0 this linear system must be a conserved system. Indeed, if any trajectories did converge to 0, this would contradict that the forward mapping is an isometry on $\mathbb{A}$. By Lemma 14 we have that all the ω-limit sets are tori. ☐

C. Nonexpansive polyhedral norms

We provide a self-contained proof that for *(real-)analytic* vector fields which are non-expansive with respect to a norm, we have a stronger convergence result. The following is essentially Theorem 21 from [10], but certain technical details were missing in the proof in that paper.

Theorem 2. *[10] Suppose we have a system $\dot{x} = f(x)$ where $f(x)$ is analytic and has bounded trajectories. Suppose the system is non-expansive with respect to some polyhedral norm. Then the solutions of the system converge to the equilibria set.*

Proof. One can show that $\|f(x(t))\|$ is nonincreasing along any trajectory, because $(d/dt)f(x(t)) = \mathcal{J}_f(x)f(x(t))$ and the logarithmic norm of $\mathcal{J}_f(x)$ is nonpositive. It follows by the LaSalle's Invariance Principle that every solution approaches a set $Z_c := \{x_0 \,|\, \|f(\phi_t(x_0))\| \equiv c\}$ for some $c \geq 0$.

We claim that any such set Z_c consists solely of equilibria. Pick any point $x_0 \in Z$ and the corresponding trajectory $x(t) = \phi_t(x_0)$. By definition of Z_c, $x(t) \in Z_c$ for all $t \geq 0$. We claim that $c = 0$, i.e. that $x(t) \equiv x_0$, so that x_0 is an equilibrium. Indeed, suppose that $c \neq 0$. Then $f(x(t))$ is always a point on a norm ball of a constant (nonzero) size. Thus it must spend a finite time on a face of this ball of constant size. Suppose that this face has normal vector η. Then $\eta \cdot f(x(t))$ will be a constant value, for a set of times in a set of positive measure, and so must be a constant value for all time, by analyticity. This implies that $\eta \cdot f(x(t))$ has this constant value for all $t \geq 0$, forcing the velocity vector $f(x(t))$ to always point in a certain direction (i.e., along the direction of η), forcing the trajectory to be unbounded, a contradiction. ☐

D. Nonexpansive maps on $\mathbb{R}^2$

In the special case of $\mathbb{R}^2$, we have some stronger results. In the following, we do not assume the norm on $\mathbb{R}^2$ is strictly convex.

Lemma 15. *The only norms preserved by a nontrivial one parameter family of linear isometries of the form e^{Bt} are the weighted ℓ^2 norms.*

Proof. To prove this we will show that whatever the norm is, it must have the same unit norm ball as that of a weighted ℓ^2 norm. Consider all possible bounded trajectories of the form $e^{Bt}x_0$. These correspond to trajectories of the linear system $\dot{x} = Bx$. Note that all the eigenvalues of B must have 0 real part, otherwise we would have points converging to 0 or diverging to ∞, contradicting that e^{Bt} should be an isometry for all t.

By Lemma 12 that there exists a matrix P such that $\frac{d}{dt}(x^\top Px) = 0$. Let C be the set $x^\top Px = 1$. Pick any $x_0 \in C$. Note that for $t \geq 0$ we have that $e^{Bt}x_0$ traces out exactly the set C. Since we assume e^{Bt} is always an isometry, we get that all points on C have the same norm (since scaling does not impact anything, we will assume the points on C have norm 1). Note that C is exactly the norm unit ball of our system's norm (indeed if any other point $x \notin C$ had norm 1, we could multiply C by a scalar to conclude x must also have norm different than 1, a contradiction), but it is also the unit ball of a weighted ℓ^2 norm. Since the unit balls of a norm determine the norm, we have in fact our norm must be a weighted ℓ^2 norm. ☐

Note that the previous lemma is not true in $\mathbb{R}^n$ for $n > 2$. For example, take $n = 3$ and consider the norm $\|(x, y, z)\| = \sqrt{x^2 + y^2} + |z|$. This norm is preserved by the linear system $\dot{x} = -y, \dot{y} = x, \dot{z} = 0$.

Lemma 16. *If a global attractor $\mathbb{A}$ contains a limit cycle, the only norm we can preserve on $\mathbb{A}$ is a weighted ℓ^2 norm.*

Proof. Suppose we have a limit cycle, and let I be the limit cycle with its interior. Then since ϕ_t takes I to itself for all time, so ϕ_t must be an isometry on I by Lemma 2. It contains an open set so by Mankiewicz's Theorem the map ϕ_t restricted to the interior of I can be extended to an affine map F_t. Thus by Lemma 15 the preserved norm must be a weighted ℓ^2 norm. □

Lemma 17. *If an ω limit set of a point x contains an equilibrium point p, then $\phi_t(x)$ converges to p.*

Proof. If p is an equilibrium point in the ω-limit set of a point x, then for every $\epsilon > 0$ we can find $t > 0$ such that $\|p - \phi_t(x)\| < \epsilon$. Since the system under consideration is non-expansive, we have that $\|p - \phi_t(x)\| < \epsilon$ for all $t > T$. Thus the trajectory is simply converging to p. □

Lemma 18. *If a system is non-expansive for a norm which is not a weighted ℓ^2 norm, then all bounded trajectories must converge to the equilibria set.*

Proof. Suppose the system is non-expansive for a norm which is not a weighted ℓ^2 norm. By Lemma 16 the system cannot have any limit cycles. By the Poincare-Bendixson theorem any ω-limit set of a point x that is not a limit cycle must contain some fixed point p, but by Lemma 17 the point p is the only fixed point of $\omega(x)$ (otherwise $\phi_t(x)$ would have to converge to two different points, which is impossible). Thus we must converge to the fixed point p. □

IV. A NECESSARY AND SUFFICIENT CONDITION FOR NONEXPANSIVITY

Here we will provide a necessary and sufficient description of nonexpansivity with respect to a norm. This condition is connected to the supporting hyperplanes of a unit ball of said norm. This can be seen as a type of Demidovich condition for contractivity [19].

Let $B_d = \{x \in \mathbb{R}^m | \|x\| \leq d\}$. For all $v \in \mathbb{R}^m$ let N_v be the set of all possible normal vectors of hyperplanes that support $B_{\|v\|}$ at v and are orientated toward the complement of $B_{\|v\|}$. In other words, let H_v be the set of hyperplanes that support $B_{\|v\|}$ at v. Then

$$N_v = \{n \in \mathbb{R}^n | \exists H \in H_v (\forall h \in H[n \cdot (h - v) = 0, \, n \cdot v \geq 0])\}.$$

In the following let $\mathbb{X} = \mathbb{R}^m$.

Theorem 3. *Suppose we have a dynamical system $\dot{x} = f(x)$ and a norm $\|.\|$ on the state space $\mathbb{X}$ of the system. Then the system is non-expansive iff for all $x \in \mathbb{X}$ and all $v \in \mathbb{R}^m$ then whenever $n \in N_v$ we must have*

$$n^\top \mathcal{J}_f(x) v \leq 0.$$

Proof. Suppose that the system is non-expansive. Then for all $t \geq 0$ we have that $\|\phi_t(x) - \phi_t(y)\| \leq \|x - y\|$. Thus we must have that for $n \in N_{x-y}$ that

$$n^\top (\phi_t(x) - \phi_t(y)) \leq n^\top (x - y)$$

or

$$n^\top ((\phi_t(x) - x) - (\phi_t(y) - y)) \leq 0.$$

Note the first inequality is due to the observation that if n is the normal vector of a supporting hyperplane H of a convex figure, and for $v \in H$ we have $n^\top v = c$, then for all points in the convex figure we must have $n^\top v \leq c$.

Now we have that $\phi_t(x) - x = tf(x) + t\epsilon_1(t)$ where $\epsilon_1(t) \to 0$ as $t \to 0$, and similarly $\phi_t(y) - y = tf(y) + t\epsilon_2(t)$ where $\epsilon_2(t) \to 0$ as $t \to 0$. Thus we get

$$n^\top (tf(x) + t\epsilon_1(t) - (tf(y) + t\epsilon_2(t)) \\ = n^\top (t(f(x) - f(y)) + t(\epsilon_1(t) - \epsilon_2(t))) \leq 0$$

or

$$n^\top ((f(x) - f(y)) + (\epsilon_1(t) - \epsilon_2(t))) \leq 0.$$

Let $t \to 0$ we get that $n^\top (f(x) - f(y)) \leq 0$. Now we have by the mean value theorem that for some real number s where $0 \leq s \leq 1$ we have that

$$n^\top (f(x) - f(y)) = \int_0^1 n^\top (\mathcal{J}_f(y + (x - y)t)(x - y))dt \\ = n^\top (\mathcal{J}_f(y + (x - y)s)(x - y)).$$

Thus also have that

$$n^\top (f(x) - f(y)) \leq 0 \\ n^\top (\mathcal{J}_f(y + (x - y)s)(x - y) \leq 0$$

Let $x_r = y + (x - y)r$, so that $x_1 = x$ and $x_r - y = (x - y)r$. We can divide by r to get the inequality

$$n^\top (\mathcal{J}_f(y + (x - y)rs)(x - y) \leq 0$$

Letting $r \to 0$ we get that $(x - y)rs \to 0$ and so we have that

$$n^\top \mathcal{J}_f(y)(x - y) \leq 0.$$

This is the desired condition. Now we will prove the other direction. Again using the mean value theorem note that

$$n^\top (f(x) - f(y)) = \int_0^1 n^\top (\mathcal{J}_f(y + (x - y)t)(x - y))dt \\ = n^\top (\mathcal{J}_f(y + (x - y)s)(x - y)).$$

Thus we have that

$$n^\top (f(x) - f(y)) = n^\top (\mathcal{J}_f(y + (x - y)s)(x - y)) \leq 0.$$

The last inequality is by assumption. Thus $n^\top (\dot{x} - \dot{y}) = n^\top (f(x) - f(y)) \leq 0$. From this it follows that the vector $x(t) - y(t)$ is not moving outside of the ball $B_{\|x-y\|}$ and so the system is non-expansive. □

A. Examples

1) Systems non-expansive with respect to the ℓ^4 norm: Using Theorem 3 we can show that there exists systems with non-expansive ℓ^p norms for $p \neq 1, 2, \infty$. For the ℓ^4 norm in $\mathbb{R}^2$ the condition from Theorem 3 is that

$$[u^3, v^3] \mathcal{J}_f(x) \begin{bmatrix} u \\ v \end{bmatrix} \leq 0.$$

Note that

$$-(u^2+2cuv-2c^2v^2)^2 = -u^4-4cu^3v+8c^3uv^3-4c^4v^4 = [u^3, v^3]\begin{bmatrix}-1 & -4c \\ 8c^3 & -4c^4\end{bmatrix}\begin{bmatrix}u \\ v\end{bmatrix} \le 0.$$

This implies that for the matrix

$$A_c = \begin{bmatrix}-1 & -4c \\ 8c^3 & -4c^4\end{bmatrix}.$$

that the linear system $\dot{x} = A_c x$ in non-expansive with respect to the ℓ^4 norm for all real numbers c. Note also that for $u = (1+\sqrt{3})cv$ we have that

$$[u^3, v^3]\begin{bmatrix}-1 & -4c \\ 8c^3 & -4c^4\end{bmatrix}\begin{bmatrix}u \\ v\end{bmatrix} = 0.$$

2) A globally convergent Hurwitz everywhere system which is not contractive with respect to any norm: We can also show that in fact there is a Hurwitz everywhere system that is globally convergent which is not non-expansive with respect to any norm. Here by *globally convergent* we mean all trajectories converge to the origin, and by *Hurwitz everywhere* we mean that the Jacobian at each point is such that all its eigenvalues have strictly negative real part. Consider the following system:

$$\dot{x} = -x \quad \dot{y} = -(x^2+1)y.$$

Now by Theorem 3 for the system to be non-expansive with respect to a norm we must have that $n^\top \mathcal{J}_f(x,y)v \le 0$ as in the notation of the theorem. Note that for the system under consideration:

$$\mathcal{J}_f(x,y) = \begin{bmatrix}-1 & 0 \\ -2xy & -(x^2+1)\end{bmatrix}.$$

For v with nonzero x coordinate we have that $\mathcal{J}_f(x,y)v$ contains every vector with a negative x coordinate. This forces n to be the vector $[-1, 0]$ or some positive multiple of this vector. There does not exist a bounded symmetric convex shape centered at the origin in $\mathbb{R}^2$ such that any supporting hyperplanes at points with nonzero x coordinate have normal $[-1, 0]$ (the only such shape with this property would be two parallel lines).

V. Conclusions

We characterized the ω-limit sets of (generally nonlinear) non-expansive dynamical systems with respect to strictly convex norms as attractors of linear systems. A common theme throughout our paper is that the isometry group of a norm is closely tied to the behavior of dynamical systems non-expansive with respect to the norm. We also provided a complete description of non-expansive systems in $\mathbb{R}^2$, and presented a Demidovich type condition which we used to provide some examples of non-expansive systems. An open question remains regarding Theorem 1 regarding a relaxation to non-strictly convex norms.

Open Problem. *Does Theorem 1 hold if the qualifier "strictly convex" is removed?*

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