LINEAR SYSTEMS OVER COMMUTATIVE RINGS: A (PARTIAL) UPDATED SURVEY

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Abstract : Some recent developments in the theory of linear systems over rings are described. The focus is on problems of regulation, as well as on applications to delay systems and computational methods for classical linear systems.

Key words: System Theory; linear systems

1. INTRODUCTION

This paper is intended as a partial update of the survey in Sontag(1976). The 1976 paper, to be referred in what follows as SSR, described the area of systems over rings and surveyed the results known at that time. Since then, a large number of further results have been obtained; it would be impossible to review all of them here. We have chosen to restrict ourselves to a short overview of parts of the area, especially regarding problems of regulation. Few of the references and results quoted in SSR will be repeated. To make the paper self-contained, however, and to emphasize application areas, we shall briefly describe again some of the motivations and early work. Later we shall treat "pole shifting" problems, dynamic regulation and its relationship to matrix fraction decompositions of transfer matrices, and the case of "principal ideal domains", where all concepts and results are easier to understand. Two other expositions, those of Kamen(1978) and Hazewinkel(1979) provide general views on subjects related to systems over rings; there is very little overlap among all three papers, and the interested reader should consult those references for aspects not treated here.

Motivation

A great deal of the theory of finite-dimensional linear systems can be derived as algebraic properties of a matrix triple (F,G,H) and its relations to various types of algebraic "input/output" objects (transfer matrices, impulse responses). It is known in fact how to carry out this theory when the entries of the matrices F, G, H belong to an arbitrary field R of coefficients. Classically, one uses R = real or complex numbers; in coding applications one uses R = finite field. Familiarity with the classical theory will be assumed in what follows. Assume now that R is a commutative ring, not necessarily a field (in other words, no division is allowed inside R). Most questions about linear systems, like those pertaining to realizations of i/o maps, or the existence of feedback controllers, can often be posed in meaningful ways in this more general context. However, generalizing from the field case is far from trivial. This is because the area of linear algebra over rings has only recently been the object of study, and because many fundamental results are simply false over rings (most particularly, those dealing with matrix diagonalization and canonical forms). Still, it turns out that many system theoretic results can be proved over rings, provided that one does not insist in using the same methodology as for fields. (In studying alternative approaches, one of course gets insight also into the "classical" case.)

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Systems over rings are of interest for various applied reasons. For example, a controlled delay-differential equation

$$(dx/dt)(t) = x(t)+3x(t-1)-2x(t-3)+u(t)$$

can be thought of as an equation

$$(dx/dt)(t) = (1+3p-2p^3)x(t) + u(t),$$

where p denotes a delay operator. The resulting equation resembles a classical finite dimensional controlled system, whose coefficients belong now to a polynomial ring $\mathbb{R}[p]$. When there are incommensurable delays in the original equations, one includes various delay operators, $p_1,\dots, p_n$, and there results a polynomial ring $\mathbb{R}[p_1,\dots,p_n]$; a complex polynomial ring is used if one wants to allow for complex coefficients in the original delay equation. One may study all this over a field of rational functions, but that has the disadvantage that synthesis procedures will in general involve inverses of p, i.e. nonrealizable ideal predictors. It is very natural to view these as systems over rings, and to apply the relevant theory. Such an approach was emphasized by E. Kamen ca. 1972 (see Kamen(1975)). The ring structure was recognized independently by Williams and Zakian(1977), but no use of systems over rings results was made by them.

A rather different motivation for studying systems over rings comes from applications to discrete time systems over the ring of integers. These are controlled linear systems where the input, state, and output values, as well as the system coefficients, are integral. Variations of this are systems over residue rings. A linear system over $\mathbb{Z}_n$, $n = 2^r$, and r = word length of a computer, is a much better model of a (fixed-point) digitally implemented system that a system over the reals, in that overflows and underflows become explicit in the system description. Systems over the integers were the original motivation for the work on realization over rings in Rouchaleau et al.(1972), as described in SSR.

Other areas of application were described in SSR. One of them, mentioned only briefly there, was that of "2D" or "image processing" systems. The work mentioned in SSR was explained in some more detail in Sontag(1978), and was carried out in much further depth by Eising(1979,1980).

One area of application that was not described in SSR was that of families of linear systems. Here one considers a parametrized set of real or complex coefficient systems

$$(*) \qquad (dx/dt)(t)=F_p x(t)+G_p u(t), \; y(t)=H_p x(t),$$

(or the corresponding discrete time version,) where p denotes a vector parameter. The study of families of linear systems, with an emphasis on the algebro-geometric global structure, was initiated by Hazewinkel and Kalman(1976). Given a system over a ring, it is natural to view it as a family of systems over residue fields; this is the "local method" used in SSR to establish some results. However, SSR did not use at all the global geometric structure which is natural for families of systems. Byrnes(1979) noticed that, when R is a ring of polynomials, (and hence one has a family of classical linear systems,) the geometric global structure can be usefully employed. He proved various results for systems over these rings using facts about families of systems obtained by various authors. Recently, Hazewinkel(1979) argued that the (global) "family of systems" approach can be useful even for rather general -nonpolynomial- rings, and showed how to derive proofs of some results using also these methods. It is very possible that these methods will become useful in the future, but they will not be explained further here. In any case, we are interested here in families of systems not as a tool in studying systems over rings, but, quite the contrary, as an area of application.

There are many reasons for studying families of systems like in (*). Given a synthesis problem, say that of feedback stabilization, it is mathematically natural to ask whether this construction can be parametrized also by p in the same way as (*) is. For example, if in (*) the parameters p appear polynomially, then one would want to know if it is possible to obtain a feedback matrix K_p dependent also polynomially on p such that, for all p, $F_p-G_pK_p$ has all its eigenvalues in a given half plane. We believe that these arguments will become eventually very useful in dealing with certain computational issues in control. In many situations, the general form of a family is a priori known, except for the precise values of some parameters. This happens for example when one takes a nonlinear system and studies linearizations around various possible operating points, or in some adaptive control situations. The methods of systems over rings allow one to place most of the computational emphasis in the offline computations involved in obtaining a regulator depending on the parameters p: the only computations needed online (when the parameters are identified) are polynomial evaluations.

2. SOME RECENT RESULTS

The topic of realization was discussed in some detail in SSR. The problem there is to study possible "internal" representations (F,G,H) of an "impulse reponse" sequence $(A_1, A_2, \ldots)$, where the A_i are $m \times p$ matrices over the commutative ring R. By realization, one means that (F,G,H) is a triple of matrices such that $A_i = HF^{i-1}G$ for all i, as in the classical case. More generally, one may allow an arbitrary R--module X as state--space, with $F: X \dashrightarrow X$, $G: R^m \dashrightarrow X$, and $H: X \dashrightarrow R^p$ all R--linear. This was discussed in SSR. For many problems it is necessary to study realizations for which X is a projective module; these will be called projective realizations. Over many rings of system-theoretic interest, projective = free, so that in that case "projective realization" can be replaced by "triple of matrices (F,G,H)". This is valid in particular if R is a ring of polynomials over a field, or the ring of real rational functions with no real poles. The notion of "projective" is the correct one for the general statements and proofs, however, since the equality projective = free is not valid for general rings, and is in any case nontrivial to establish. (In geometric language, one first constructs vector bundles, and then uses other results to conclude that they are trivial.)

Further realization studies, not mentioned in SSR, have dealt mainly with specific types of rings, like those of interest for delay systems. For example, Cliff and Burns(1978) studied special realizations over single variable polynomial rings, and Eising and Hautus(1978) gave an alternative algorithm for realizations over principal ideal domains (of which single variable polynomial rings are one of the main examples). For more general polynomial rings, the results of Rouchaleau and Sontag(1978) apply to show that minimal realizations of dimension equal to the McMillan degree exist when dealing with polynomial rings in two variables, but not in the three or more variables case. On a related area, Kamen(1979) based his realization theory for linear time varying systems on ideas very analogous to those used for systems over rings.

Matrix fractions

Probably the most important recent developments have dealt with the study of matrix fraction descriptions for systems over rings.

A (proper or causal) transfer matrix over the ring R is a matrix $W = (p_{ij}(z)/q_{ij}(z))$ of rational functions with all q_{ij} monic and the degree of q_{ij} greater or equal to that of p_{ij}. Strict causality correponds to these degrees being strictly greater. Via an expansion

$$W(z) = \sum A_i z^{-i} ,$$

transfer matrices correspond to impulse responses, with $A_0 = 0$ for strict causality. The realization requirement: $A_i = HF^{i-1}G$ for all i, is equivalent to the equality $W(z) = H(zI-F)^{-1}G$. In the nonstrictly causal case one adds a feedforward term, as usual. See SSR for more details on the general relationship between transfer matrices and realizations.

A right (resp., symmetric) polynomial matrix fraction decomposition of a transfer matrix W is given by a pair of matrices (P,Q) (resp., three matrices (P,Q,R)) over R[z] such that $W = PQ^{-1}$ (resp., $PQ^{-1}R$). One requires also that the leading coefficient of the determinant of Q be invertible in R. A left decomposition is one for which $W = Q^{-1}P$. It is well known in the theory over fields that these types of decomposition are very useful in studying regulation problems. Recently, Khargonekar (see Khargonekar(1980) and Khargonekar and Emre(1980)), and independently Conte and Perdon(1980), have derived results which clarify the relations between the existence of various types of matrix fraction decompositions and the existence of different classes of internal realizations of systems over rings. The results of Conte and Perdon hold over principal ideal domains, and those of Khargonekar over arbitrary commutative rings, and are based on rather deep algebraic results.

In order to explain the connections between matrix fractions and realizations, we need some more terminology. A pair (P,Q) is right coprime if whenever T is a common right factor of P and Q, i.e. $P = P_oT$ and $Q = Q_oT$, then T has to be unimodular. A similar definition holds for left coprimeness.

The results of Conte and Perdon for principal ideal domains, and the results of Khargonekar for arbitrary commutative rings with identity, prove that projective (in the case of p.i.d.'s free) strongly observable realizations are in a one to one correspondence with left representations $Q^{-1}P$; the corresponding realization has minimal dimension if and only if P and Q are left coprime. Dually, (free) reachable realizations correspond to right factorizations PQ^{-1}, with right coprimeness corresponding to realizations being canonical. (These one to one correspondences all hold modulo unimodular common factors of P, Q, and modulo system isomorphisms.) It was shown further by Khargonekar that one also has a one-to-one correspondence between projective realizations and symmetric matrix descriptions (modulo "strict system equivalence").

Recall from SSR that a (projective) realization is split if (F,G) and (F',H') are both reachable; equivalently, if the matrices (zI-F,G) and (zI-F',H') both admit right inverses over the ring R[z]. Khargonekar proved that an input/output map admits a split realization if and only if it admits a Bezout right polynomial matrix factorization, or equivalently, if it admits a Bezout left one. A Bezout right decomposition $W = PQ^{-1}$ is one for which there exist polynomial matrices A and B such that

$$AP + BQ = I ;$$

similarly for left Bezout. Of course a Bezout right (resp., left) factorization is also right (resp., left) coprime, but the converse is in general false (except in the "classical" case).

For a polynomial ring $R = \mathbb{R}[p_1,\ldots,p_r]$, reachability of (F,G) is equivalent to $(F(p^*),G(p^*))$ being reachable for all particular values $p = p^*$ over the complexes. Thus a reachable system over R corresponds to a family of linear systems each of which is reachable. This gives an interpretation to the notion of split realization: each system $(F(p^*), G(p^*), H(p^*))$ must be canonical in the usual finite dimensional sense. The Hankel matrix criterion in Sontag(1978a) can be used to determine if a transfer matrix admits a split realization, using the impulse response parameters. A recent result of Lee and Olbrot(1980) states that reachability is generic for systems over a polynomial ring as long as the number of input channels is larger than r. Using also the dual version, this implies for instance for the case r=1 that systems with at least two control and measurement channels are "in general" split systems. Here "generically" means the following: for each n and k, restrict attention to all those dimension-n systems whose coefficients are polynomials in p of degree (at most) k. Let v be a vector listing all the coefficients of all the polynomials appearing; "generic" then means "containing an open dense set" in the space of all these vectors v. Since systems over rings of single variable polynomials are good models of many delay differential situations, it appears that the "split" condition is somewhat less restrictive than it may seem.

For various reasons in regulation problems, one is interested also in studying matrix factorizations into "stable" rational matrices. The notion of stability is introduced in this abstract context by postulating the existence of a subset of "Hurwitz" or stable polynomials S of R[z]; this set is assumed to consist of monic polynomials and to be closed under multiplication. For a given such set S, a "stable transfer matrix" is one all whose entries are stable rational functions p_{ij}/q_{ij} with q_{ij} stable. A "stable system" is one for which the characteristic polynomial of F is in the stability set S. (In applications to delay systems, -say with r=1- one takes for S the set of polynomials with no roots (r,z) with $r = e^{-cz}$ and real part of z to the left of some given number, and c = length of delay. For families of systems, ask that each polynomial q have no unstable roots for each value of the parameters.) Also with respect to a fixed stability set, one defines the concept of a stabilizable system. This is a (free) system for which $(zI-F,G)$ is right invertible over the ring of stable transfer functions: equivalently, the minors of this matrix must generate a unit ideal over that ring. Similarly for $(zI-F',H')$ and detectability. These notions are introduced in Hautus and Sontag(1979), where spectral equivalents of the above conditions are also discussed. One may interpret stabilizability (and dually, detectability,) via the asymptotic controllability of the original system (see Khargonekar and Sontag(1981)).

Assume given a stability set S as above. A right (stable) rational factorization of a transfer matrix W is given by a pair of stable rational matrices P, Q, with $W = PQ^{-1}$. This factorization is called (stable-) Bezout if there are stable rational matrices A and B such that $AP+BQ=I$. It is proved in Khargonekar and Sontag(1981) -subject to the condition that projective = free over the ring- that a transfer matrix admits such a factorization if and only if the corresponding i/o map admits a realization which is free, reachable, and detectable. It is easier to understand these results if one has a transfer matrix which admits a realization with G = identity: in that case, $W = H(zI-F)^{-1}$ and the detectability condition translates exactly into the Bezout property.

Regulation

The problem (over rings) of modifying by feedback the characteristic polynomial of a reachable pair (F,G) was discussed in SSR. It was seen there that it is in general impossible to apply the arguments ("Heymann's lemma",or canonical forms), used in the classical case. Using an algebraic-geometric approach, Byrnes(1978) showed that the argument can be applied, for R = a polynomial ring, if one makes the extra assumption that the Kronecker indexes remain constant over all linear systems obtained by specialization of the parameter. This condition is analogous to the "index invariance" conditions used for linear time varying systems and in dealing with the local feedback linearization of nonlinear systems. One can extend the result to any ring over which projective = free, following the method used by Lee et al.(1980) for systems over principal ideal domains. A paper of Wyman(1978) studied homological aspects of these problems and showed that, in a very precise sense, the problem of coefficient assignment (of the characteristic polynomial of F by feedback) is dual to the problem of reachability. The questions of whether coefficient assignment is possible over principal ideal domains, and of whether pole assignment is possible over rings of more than one variable (extending results of Morse for principal ideal domains -see SSR) were open until recently: they are both answered in the negative in Bumby et al.(1981). All these questions of feedback by a matrix over the same ring have turned out to be very difficult: even in the case of polynomial rings, it is open whether stabilizability of the individual systems suffices to insure stabilizability by a polynomial feedback, or to give a constructive procedure to achieve stabilization in the reachable case.

An alternative to using "constant" feedback is to use dynamic feedback. Note that in fact a "constant" feedback is not so, since the elements of the feedback matrix K may contain "memory" in some applications (e.g., delay systems). Especially under digital control, there is no reason to restrict attention to feedback over R: it is just as simple to implement a transfer matrix in the feedback loop. Moreover, when the state feedback problem is seen as just one step in the complete (input/output) regulator construction, a dynamic feedback will be implemented in any case. It was proved in Hautus and Sontag(1979) that, over any ring and for any given element "a" in R, reachable systems can always be made part of a feedback system with characteristic polynomial a power of $(z-a)$; this gives stabilization with arbitrary convergence rates. A much stronger result was obtained by Emre and Khargonekar(1980), who showed that the characteristic polynomial of the closed loop system (plant + feedback compensator) can be assigned arbitrarily, provided a certain minimal dimension be allowed for the feedback compensator. In fact, they prove that a complete regulator construction is possible, including an observer, when the system is split. In this case, the characteristic polynomial of the overall closed loop system is the product of two polynomials -one factor representing dynamic state feedback as explained above and the other factor representing the characteristic polynomial of the observer. This resembles the "separation principle" of the classical case. This should be compared with the methods based on matrix fractions -see for instance Desoer et al.(1980)- which give an input/output version for transfer matrices admitting a Bezout factorization (the "polynomial" case of the setup of Desoer et al.). The i/o results provide stability, but not pole assignment. The relationship between the i/o and state approaches follows from the results mentioned before.

For systems which are only reachable and detectable, not necessarily split, Emre and Khargonekar show how to combine their state feedback construction with the results on observers of Hautus and Sontag(1979) in order to achieve regulator synthesis (the observer dynamics are not arbitrary in this case). This corresponds in the input/output sense to having a stable Bezout rational factorization. This compares to the "rational" case of the setup of Desoer et al.(1980) where the rational Bezout condition (plus one more condition) is used to achieve regulation: the correspondence is worked out in detail in Khargonekar and Sontag(1981). Finally, Emre(1981) has now proved that a system which is (just) stabilizable and detectable admits a regulator: arbitrary assignment of closed loop input/output dynamics (as in the reachable and detectable case) is not possible anymore, as one should expect.

A special case

We summarize here some results on regulation for the case of a principal ideal domain R and stability set S. The statements are especially simple in that case. Proofs are given in Khargonekar and Sontag(1981).

The following facts are equivalent for an i/o map f and its transfer matrix W:

(a) f has a reachable detectable realization.
(b) f has a stabilizable detectable realization.
(c) The canonical realization of f is reachable detectable.
(d) The canonical realization of f is stabilizable detectable.
(e) W has a stable Bezout rational factorization.
(f) f has a regulable realization.

(Here "regulable" means that there is an input/output regulator which makes the corresponding close loop system stable. This implies all reasonable notions of i/o regulation for f.)

Two interesting examples of the above are delay systems and families of linear systems over a field. The delay case is somewhat complicated to explain in detail here, because of difficulties in determining precisely the spectrum of the ring of stable transfer functions: see Hautus and Sontag(1979). For families of real systems one has S = the set of real polynomials in (p,z), monic in z, with no roots z with $re(z) \geq a$ (a = degree of stability), for any real p. (For discrete time, $|z| \geq a$.) Then (F,G,H) is reachable if and only if

$(F(p^*),G(p^*))$ is reachable for all p^* complex,

and it is detectable if and only if

$(F(p^*),H(p^*))$ is detectable for all real p^*.

3. OTHER RESULTS

Reasons of space preclude us from treating some recent results on optimal regulation for systems over rings. In these results, the ring in question is assumed to have a normed structure so that optimization problems can be posed. For example, for image processing one may want to consider infinite vectors of (columns of) images as belonging to an l_2 space, with the ring operators corresponding to a column by column processing. The type of question asked here has to do with the possibility of solving problems of regulation in the same ring. The papers of Kamen(1979) and of Byrnes(1980), and the references therein, should be consulted in this regard. Some questions of optimal filtering for delay systems, using systems over rings, were studied by Duncan(1979).

A great deal of problems remain open. Furthermore, even in those cases in which one does have theoretical results, the methods are far from easily implementable. In principle, one may use methods from elimination theory in order to carry out many of the constructions involved in regulator synthesis, when working with polynomial rings. However, such methods are computationally inefficient, except in the single variable case. The development of good algorithms is an important goal to be attained in order for the theory of systems over rings to become a useful tool in practical system design: since most of the procedures are basically constructive, it should be possible to achieve this goal in the near future.

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