

# Stability and Stabilization of Piecewise Affine and Hybrid Systems: An LMI Approach

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## Abstract

In this paper we present various algorithms both for stability analysis and state-feedback design for discrete-time piecewise affine systems. As in [9], our approach hinges on the use of piecewise quadratic Lyapunov functions that can be computed as the solution of a set of linear matrix inequalities. We show that the continuity of the Lyapunov function is not required in the discrete-time case. Moreover, the basic algorithms are made less conservative by exploiting the switching structure of piecewise affine systems and by using relaxation procedures.

**Keywords:** piecewise affine systems, hybrid systems, stability

## 1 Introduction

Hybrid systems have recently attracted the interest of the control and computer science communities, due to their ability to model the interaction between continuous dynamics and logic components. The analysis and control methodologies for hybrid systems strongly depend on the modeling framework adopted. It is known that a rich class of hybrid models can be described via Piecewise Affine (PWA) systems that are defined by partitioning the state-space in a finite number of polyhedral regions and associating to each region a different linear dynamic model. Sontag [13] showed that PWA systems and interconnections of linear systems and finite automata are equivalent. In [1] it is proved, by using a constructive argument, that the class of PWA and discrete-time hybrid systems in the Mixed Logic Dynamical (MLD) form [2] coincide. The expressivity of the MLD form is wide enough to represent a large class of hybrid systems, for instance linear hybrid dynamical systems, hybrid automata, some classes of discrete-event systems, and systems with qualitative inputs/outputs [2, 1]. The equivalence theorem (that is recalled in Section 2) allows to explicitly describe all these models in the PWA form.

The MLD/PWA framework is also useful both in investigating structural properties of hybrid systems such as observability, controllability [1] and in designing Model Predictive Control schemes [2], state-estimation algorithms and verification procedures [3].

The first aim of this paper is to develop tests for checking the stability of the origin of a PWA system. The main motivation for this research is that there is no easy way to check the stability even of an autonomous PWA systems with two polyhedral regions. Indeed, Blondel and Tsitsiklis [5] showed that in general this problem is either  $\mathcal{NP}$ -complete or undecidable. Moreover, it is also hopeless to deduce the stability/instability of a PWA system from the stability/instability of its components subsystems [6]. These results highlight that, in order to derive practical stability tests, one must either resort to a restricted subclass of PWA system or look for *sufficient* conditions. A first algorithm to test stability for discrete-time PWA systems was proposed in [3] by exploiting verification and robust simulation ideas. However these conditions apply only to PWA systems where the origin is strictly contained in one region. Nevertheless the method in [3] can be viewed as complementary to the tests proposed in this paper.

In this work we were largely inspired by the research of Johansson and Rantzer [9] where the stability of a continuous-time PWA system was investigated by looking for piecewise quadratic Lyapunov functions computed by solving Linear Matrix Inequalities (LMIs). This approach obviously leads to sufficient conditions, i.e. if the LMIs have a solution, the origin of the system can be classified as asymptotically stable but nothing can be said if no solution is found. We show that similar LMI based procedures can be fruitfully exploited for discrete-time PWA systems both for stability analysis and state-feedback design. However there are two major differences between the continuous and the discrete-time case. First, in the former only continuous Lyapunov functions can be used whereas in

the latter discontinuous Lyapunov functions are also allowed. This obviously increases the flexibility of the method. Second, in discrete-time, switching can also occur between non-adjacent regions and this fact must be properly handled in the analysis/synthesis algorithms.

We propose various stability tests and controller synthesis procedures that exhibit different degrees of flexibility. In Section 3 we review the basic results in stability analysis for PWA systems by using a common quadratic Lyapunov function. In this case we also show how to design a piecewise linear state-feedback. We generalize these results in Section 4 by considering piecewise quadratic Lyapunov functions. Here, the switching between non-adjacent regions can increase the complexity of the algorithms considerably and we discuss how to partially overcome this problem by avoiding to consider switches the system cannot exhibit. In Section 5 we propose a relaxation strategy for the LMIs involved in the stability analysis that leads to the most powerful test we propose. This relaxation is inspired by the work [9]. Finally, we discuss the degree of conservativeness of the analysis algorithms proposed (Section 6) and provide an example (Section 7). All proofs are omitted and can be found in [11].

## 2 The PWA form of Hybrid Systems

The state-space equations describing an MLD system are

$$x_{k+1} = Fx_k + G_1u_k + G_2\delta_k + G_3z_k \quad (1a)$$

$$y_k = Hx_k + D_1u_k + D_2\delta_k + D_3z_k \quad (1b)$$

$$V_2\delta_k + V_3z_k \leq V_1u_k + V_4x_k + V_5 \quad (1c)$$

where  $x \in \mathbb{R}^{n_c} \times \{0,1\}^{n_\ell}$  are the continuous and binary states,  $u \in \mathbb{R}^{m_c} \times \{0,1\}^{m_\ell}$  are the inputs,  $y \in \mathbb{R}^{p_c} \times \{0,1\}^{p_\ell}$  the outputs, and  $\delta \in \{0,1\}^{r_\ell}$ ,  $z \in \mathbb{R}^{r_c}$  represent auxiliary binary and continuous variables respectively. The auxiliary variables are introduced when translating propositional logic into linear inequalities. For a detailed description of the modeling capabilities of MLD systems, we defer to [2, 1].

Recently, in [1], it was shown that MLD system can be represented in the PWA form and vice-versa. PWA systems are described by the state-space equations:

$$\begin{aligned} x_{k+1} &= A_i x_k + B_i u_k + a_i \\ y_k &= C_i x_k + c_i \\ (x, u) &\in \mathbb{X} \end{aligned}, \text{ for } \begin{bmatrix} x_k \\ u_k \end{bmatrix} \in \mathcal{X}_i \quad (2)$$

where the state+input set  $\mathbb{X} \subset \mathbb{R}^{n_c+n_\ell} \times \mathbb{R}^{m_c+m_\ell}$  is a polyhedron containing the origin,  $\{\mathcal{X}_i\}_{i=1}^s$  is a polyhedral partition<sup>1</sup> of  $\mathbb{X}$  and  $a_i$ ,  $c_i$  are constant vectors

<sup>1</sup>Each set  $\mathcal{X}_i$  is a (not necessarily closed) convex polyhedron s.t.  $\mathcal{X}_i \cap \mathcal{X}_j = \emptyset$ ,  $\forall i \neq j$ ,  $\bigcup_{i=1}^s \mathcal{X}_i = \mathbb{X}$ .

of suitable dimension. We refer to each  $\mathcal{X}_i$  as a *cell*. In [2] it is shown that PWA systems can be represented in the MLD form. The converse is stated in the next proposition.

**Proposition 1 ([1])** *Consider generic trajectories  $x_k$ ,  $u_k$ ,  $y_k$  of the MLD system (1). Then, there exist a polyhedral partition  $\{\mathcal{X}_i\}_{i=1}^s$  of the state+input space  $\mathbb{X}$  and 5-tuples  $(A_i, B_i, C_i, f_i, g_i)$ ,  $i = 1, \dots, s$ , such that  $x_k$ ,  $u_k$ ,  $y_k$  satisfy (2).*

Proposition 1 was proved in [1] by using a constructive argument that allows the explicit computations of the sets  $\mathcal{X}_i$  and the matrices  $A_i$ ,  $B_i$ ,  $C_i$ ,  $a_i$  and  $c_i$  defining the PWA system. In this work we will assume that the PWA system is switching its dynamics depending only on the state. With a slight abuse of notation we will henceforth assume that the set  $\mathbb{X}$  is a subset of  $\mathbb{R}^{n_c+n_\ell}$ .

According to the notation in [8], we call  $\mathcal{I} = \{1, \dots, s\}$  the set of indices of the state space cells.  $\mathcal{I}$  is partitioned as  $\mathcal{I} = \mathcal{I}_0 \cup \mathcal{I}_1$ , where  $\mathcal{I}_0$  are the indices of the cells whose boundary contains the origin  $x = 0$ , and  $\mathcal{I}_1$  are the indices of the cells, whose boundary that does not contain the origin. In this paper we will focus on the stability of the origin. Then, we assume that  $x = 0$  is an equilibrium of system (1). By using Proposition 1, this implies that  $a_i = 0$ ,  $\forall i \in \mathcal{I}_0$  in the PWA form (2). Moreover, to analyze stability, we restrict our attention to *autonomous* PWA systems, and, for the regulator design, we will consider a *piecewise linear* state feedback

$$u_k = K_i x_k \quad \forall x_k \in \mathcal{X}_i. \quad (3)$$

**Definition 1** *The equilibrium state  $x = 0$  of a system  $x_{k+1} = f(x_k)$  is **stable** if, for any  $\epsilon > 0$ , there exists a  $\delta(\epsilon) > 0$ , such that  $\|x_0\| < \delta(\epsilon) \Rightarrow \|x_k\| < \epsilon$ ,  $\forall k > 0$ . If, in addition,  $\lim_{k \rightarrow +\infty} \|x_k\| = 0$  the origin is **asymptotically stable**. Finally, let  $\mathbb{X}_0 \subseteq \mathbb{X}$  such that  $0 \in \mathbb{X}_0$ . The origin  $x = 0$  is **asymptotically stable on  $\mathbb{X}_0$**  if it is asymptotically stable and for any initial state  $x_0 \in \mathbb{X}_0$ ,  $\lim_{k \rightarrow +\infty} \|x_k\| = 0$ .*

## 3 Quadratic Lyapunov Function

### 3.1 Stability Analysis

Based on Lyapunov theory arguments, the asymptotic stability of a PWA discrete-time system can be checked with a quadratic Lyapunov function

$$V(x) = x^T P x \quad (4)$$

In [10, 12] it is recalled that sufficient conditions on  $P$  for asymptotic stability are given by the LMIs:

$$P > 0 \quad (5)$$

$$A_i^T P A_i - P < 0 \quad \forall i \in \mathcal{I} \quad (6)$$

If such a  $P$  exists, the function (4) is called a common Lyapunov function for the matrices  $\{A_1, \dots, A_s\}$  and system (2) is termed *quadratically stable* (Q-stable).

### 3.2 State Feedback Synthesis

We consider the synthesis of a piecewise linear state feedback (3) for system (2) that stabilizes the origin by means of a quadratic Lyapunov function (4). In other words we look for the matrices  $P$  and  $K_i$ ,  $\forall i \in \mathcal{I}$  that satisfy the conditions

$$P > 0 \quad (7)$$

$$(A_i + B_i K_i)^T P (A_i + B_i K_i) - P < 0 \quad \forall i \in \mathcal{I} \quad (8)$$

Since both variables  $P$  and  $K_i$  are unknown, the matrix inequality (8) is apparently nonlinear. However, as for LTI discrete-time systems, it can be rewritten as an LMI by using Schur complements. Introducing new variables  $Y_i = K_i Q$  and  $Q = P^{-1}$ , we can solve for  $Y_i$  and  $Q$  instead of  $K_i$  and  $P$ . The resulting LMIs are

$$\begin{bmatrix} Q & (A_i Q + B_i Y_i)^T \\ (A_i Q + B_i Y_i)^T & Q \end{bmatrix} > 0, \quad \forall i \in \mathcal{I} \quad (9)$$

$$Q > 0 \quad (10)$$

and the state feedback vector can be recovered as

$$K_i = Y_i Q^{-1} \quad (11)$$

To what concerns the region of attraction of the state-feedback derived so far, let  $\bar{V}_1(x) = x^T P x$  and choose  $\mathbb{X}_0$  as the largest level set  $\{x \in \mathbb{X} : \bar{V}_1(x) \leq \xi, \xi > 0\}$  contained in  $\mathbb{X}$ . Then every state-trajectory of the controlled system starting from  $\mathbb{X}_0$ , besides converging to the origin, does not leave the set  $\mathbb{X}$ . In summary, we have proved the following result

**Lemma 1** *Let  $Q$  and  $Y_i$  be the solutions of the LMIs (10), (9) and  $K_i$  be computed as in (11). Then, the piecewise linear state feedback  $u_k = K_i x_k$ ,  $\forall x_k \in \mathcal{X}_i$  stabilizes asymptotically the origin of (2) on  $\mathbb{X}_0$ .*

## 4 Piecewise Quadratic Lyapunov Function

### 4.1 Stability analysis

A Piecewise Quadratic (PWQ) Lyapunov function for a PWA system can be defined as

$$V(x) = x^T P_i x \quad \forall x \in \mathcal{X}_i \quad (12)$$

As in [9] we consider PWQ Lyapunov functions, but contrary to the continuous-time case the function  $V(x)$  can be discontinuous across cell boundaries. Indeed, we do not have to require continuity of  $V(x)$  to prove stability, as long as the number of cells is finite, see [11]. For stability, it has to hold that  $V(x)$  is positive-definite in a neighborhood of the origin and that

$$\Delta V(x_k, x_{k-1}) = V(x_k) - V(x_{k-1}) < 0 \quad (13)$$

Assuming that  $x_k \in \mathcal{X}_i$  and  $x_{k-1} \in \mathcal{X}_j$ , we get the LMIs to satisfy in this case as:

$$A_j^T P_i A_j - P_j < 0 \quad \forall (j, i) \in \mathcal{S}_{all} \quad (14)$$

$$P_i > 0, \quad \forall i \in \mathcal{I} \quad (15)$$

where  $\mathcal{S}_{all} = \mathcal{I} \times \mathcal{I}$ . In [11] we show, that these conditions are also sufficient for asymptotic stability on  $\mathbb{X}_0$ , i.e. the largest level set of  $V(x)$  contained in  $\mathbb{X}$ . When the LMIs (14)-(15) are feasible, we term the PWA systems *piecewise quadratically (PWQ) stable*. Now, the main difficulty, compared to the continuous-time case, is that we have to satisfy the LMIs (14) for all the pairs  $(i, j)$  because the state may switch in one step between non adjacent cells. Without further analysis of the system we have to take into account all possible switches from each state space region to each other region. Therefore the number of possible switches grows quadratically with the number of cells. However, this condition can be relaxed because usually not all the transitions between cells  $\mathcal{X}_i$  and  $\mathcal{X}_j$  are allowed. Let  $\mathcal{S}$  be the set of all ordered pairs  $(j, i)$  of indices, denoting the possible switches from cell  $j$  to cell  $i$ :

$$\begin{aligned} \mathcal{S} &= \{(j, i) : j, i \in \mathcal{I} \text{ and } \exists k \in \mathbb{N}_0, \\ &\text{such that } x_k \in \mathcal{X}_i \text{ and } x_{k-1} \in \mathcal{X}_j\} \end{aligned}$$

The set  $\mathcal{S}$  can be determined via reachability analysis for MLD systems [1, 4]. Since for asymptotic stability it is enough that (14)-(15) holds for  $(j, i) \in \mathcal{S}$ , we have the following result.

**Theorem 1** *The origin of the PWA (2) is asymptotically stable on  $\mathbb{X}_0$  if there exist  $s$  matrices  $P_i$ , such that the following LMIs are satisfied:*

$$P_i > 0 \quad \forall i \in \mathcal{I} \quad (16)$$

$$A_j^T P_i A_j - P_j < 0 \quad \forall (j, i) \in \mathcal{S} \quad (17)$$

In this case the PWA system is termed  $\mathcal{S}$ -PWQ stable. Note that in  $\mathcal{S}$  there are in general also pairs of the form  $(i, i)$  for each cell that is not left in one step. For these cells (17) states, that  $A_i$  must be stable. This means, that we cannot show stability of a stable piecewise linear system, whose components are unstable, with a piecewise quadratic Lyapunov function (12), if the system stays in the same unstable cell for more than one time step.

There are also PWA systems where the set  $\mathcal{S}$  contains both pairs of indices  $(i, j)$  and  $(j, i)$ . We define  $\mathcal{S}_2$  as the set of such indices:

$$\mathcal{S}_2 = \{(i, j) \in \mathcal{S} : (j, i) \in \mathcal{S} \text{ and } i \neq j\} \quad (18)$$

We can state a necessary condition for the existence of a PWQ Lyapunov function (12) for systems having a nonempty set  $\mathcal{S}_2$ .

**Lemma 2** Assume that for a PWA system (2), the set  $\mathcal{S}_2$  is nonempty and contains  $2N$  elements. If a PWQ Lyapunov function (12) satisfying (16)-(17) exists, then the  $2N$  matrices  $A_{i_1}A_{j_1}, A_{j_1}A_{i_1}, \dots, A_{i_N}A_{j_N}, A_{j_N}A_{i_N}$  must have all eigenvalues inside the unit circle.

We can generalize the result of Lemma 2 to systems that can cycle across an arbitrary number of cells:

$$j_1 \rightarrow j_2 \rightarrow j_3 \rightarrow \dots \rightarrow j_n \rightarrow j_1 \quad (19)$$

In this case the existence of a PWQ Lyapunov function implies the stability of all products of matrices denoting the cells, the system can cycle through. Define the set of ordered  $n$ -tuples of indices

$$\mathcal{S}_n = \{(j_1, \dots, j_n) \in \mathcal{I}^n : \{(j_1, j_2), \dots, (j_{n-1}, j_n), (j_n, j_1)\} \subset \mathcal{S} \text{ and } j_i \text{ all different}\}$$

and define the set of products of matrices

$$\tilde{A} = \cup_{\ell=2}^{\infty} \{A : A = \prod_{\tau=1}^{\ell} A_{i_{\tau}}, (i_1, \dots, i_{\ell}) \in \mathcal{S}_{\ell}\} \quad (20)$$

We have the following corollary:

**Corollary 1** Assume that for a PWA system (2), the sets  $\mathcal{S}_n$  ( $n = 2 \dots \infty$ ) are not all empty. If a PWQ Lyapunov function (12) exists, then all matrices in  $\tilde{A}$  must have all eigenvalues inside the unit circle.

#### 4.2 State Feedback Synthesis

The same rationale used in section 3.2 applies also to the case of a piecewise quadratic function. Using the notation  $W_j = K_j P_j^{-1}$  we get  $(\forall (j, i) \in \bar{\mathcal{S}})$

$$\begin{bmatrix} P_i^{-1} & (A_j P_j^{-1} + B_j W_j) \\ (A_j P_j^{-1} + B_j W_j)^T & P_j^{-1} \end{bmatrix} > 0 \quad (21)$$

where the set  $\bar{\mathcal{S}}$  will be defined next. We still have to impose the condition

$$Q_i = P_i^{-1} > 0, \quad \forall i \in \mathcal{I} \quad (22)$$

The detailed derivation of (21) is reported in [11]. The LMIs (21), (22) provide the synthesis of a piecewise linear controller that stabilizes the origin, as can be shown with the PWQ Lyapunov function (12).

The LMI (21) must be fulfilled again for all possible pairs  $(j, i)$  corresponding to regions, the system can switch to in one step. However, the switching must be predicted in *closed loop*. Lacking of better knowledge we can always choose  $\bar{\mathcal{S}} = \mathcal{S}_{all} = \mathcal{I} \times \mathcal{I}$ . This takes into account all conceivable pairs of cells in a system with  $s$  cells. Such a brute force approach could however introduce an unnecessarily high level of conservativeness. For a particular system at hand, there

are some considerations that allow to take  $\bar{\mathcal{S}}$  as a smaller set than  $\mathcal{S}_{all}$ . Take for instance a system controlled on the high level by a finite state machine. If the finite state machine does not allow arbitrary state transitions, the corresponding PWA system will have a set  $\bar{\mathcal{S}}$ , which is smaller than  $\mathcal{S}_{all}$ . In some cases one can also know, from practical insights, that  $\bar{\mathcal{S}} = \mathcal{S}$ .

The results derived so far are summarized in the following Lemma:

**Lemma 3** Let  $\bar{V}_2(x) = x' P_i x$ ,  $\forall x \in \mathcal{X}_i$  and let  $\mathbb{X}_0$  be the largest level set of  $\bar{V}_2(x)$  contained in  $\mathcal{X}$ .

If  $\bar{\mathcal{S}}$  contains all possible pairs of indices corresponding to the cells the closed-loop PWA system can switch in one step, a stable piecewise linear state feedback that stabilizes asymptotically the origin on  $\mathbb{X}_0$  can be found solving the LMIs (21) and (22) for  $Q$  and  $Y_i$ .  $K_i$  is then given by equation  $W_j = K_j P_j^{-1}$ .

### 5 Piecewise Quadratic Lyapunov Function with Relaxations – Stability Analysis

In this Section we present another method to relax the conservativeness of the stability analysis via PWQ Lyapunov functions. These ideas have been proposed for continuous time systems in [9] and [7]. Here, we consider some relaxations on the LMIs denoting the negative or positive definiteness of the matrices of interest. Since the state space is partitioned into polyhedral cells, we can find matrices  $E_i$  and  $e_i$  such that

$$\mathcal{X}_i = \{x : \bar{E}_i \begin{pmatrix} x \\ 1 \end{pmatrix} = [E_i \ e_i] \begin{pmatrix} x \\ 1 \end{pmatrix} \geq 0\} \quad (23)$$

The inequality sign in (23) means that each entry of the vector on the left hand side has to be positive. Following the rationale of [8] we note that the LMIs (16) and (17) are actually valid on the whole state space, even though they would only be required to hold in the regions  $i$  and  $j, i$  respectively. We can remove some conservativeness if we can find matrices  $f_i$  and  $g_{j,i}$  such that

$$x^T f_i x \geq 0 \quad \text{if } x \in \mathcal{X}_i \quad (24)$$

$$x^T f_i x \leq 0 \quad \text{if } x \notin \mathcal{X}_i \quad (25)$$

$$x^T g_{i,j} x \geq 0 \quad \text{if } x \in \mathcal{X}_j \cup \mathcal{X}_i \quad (26)$$

$$x^T g_{i,j} x \leq 0 \quad \text{if } x \notin \mathcal{X}_j \cup \mathcal{X}_i \quad (27)$$

Note that we do not require the matrices  $f$  and  $g$  to be positive or negative definite, we only require that the quadratic forms take on the signs mentioned above in the corresponding regions. With these matrices the following relaxed conditions for stability can be formu-

lated,

$$P_i - f_i > 0 \quad \forall i \in \mathcal{I} \quad (28)$$

$$A_j^T P_i A_j - P_j + g_{i,j} < 0 \quad \forall (j, i) \in \mathcal{S}. \quad (29)$$

From the assumptions on  $f$  and  $g$  it follows that the fulfillment of the LMIs (28) and (29) implies the fulfillment of (16) and (17) in the regions of relevance. On the other hand, for  $x \notin \mathcal{X}_i \cup \mathcal{X}_j$ , the expressions (25) and (27), may render the LMIs (28) and (29) easier to satisfy. A possible choice for  $f$  is:

$$f_i = \bar{E}_i^T U_i \bar{E}_i, \quad (30)$$

where  $U_i$  has non-negative entries. The choice of  $g$  is more difficult however, because in general it cannot be defined in terms of the matrices  $\bar{E}_i$ , due to the switching between different regions. One possibility to circumvent all problems is:

$$g_{j,i} = \begin{cases} \bar{E}_i^T W_i \bar{E}_i & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \quad (31)$$

where  $W_i$  has non-negative entries. This avoids relaxing the LMIs (29), where two different indices for the regions  $i$  and  $j$  appear. By summarizing the previous discussion we have the following result.

**Lemma 4** *If the LMIs*

$$P_i - \bar{E}_i^T U_i \bar{E}_i > 0 \quad \forall i \in \mathcal{I} \quad (32)$$

$$A_j^T P_i A_j - P_j < 0 \quad \forall (j, i) \in \mathcal{S}, i \neq j \quad (33)$$

$$A_i^T P_i A_i - P_i + \bar{E}_i^T W_i \bar{E}_i < 0 \quad \forall (i, i) \in \mathcal{S}. \quad (34)$$

*admit a solution in  $P$ ,  $U_i$  and  $W_i$ , the origin of (2) is asymptotically stable on  $\mathbb{X}_0$  and the PWA system is termed  $\mathcal{S}$ -PWQ stable with relaxations.*

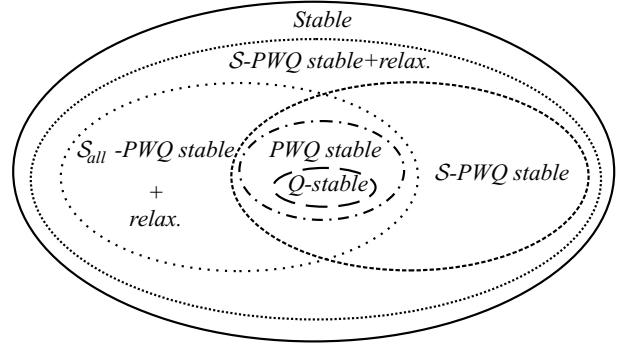
To prove this result it is enough to note that the LMIs denoting the decrease from region  $j$  to region  $i$  are not relaxed. Therefore, Lemma 4 is a consequence of Theorem 1.

**Remark 1** As in [9] one can impose some structure to the matrices  $U_i$  and  $W_i$ , for instance by taking them diagonal or lower triangular. This decreases the flexibility of the relaxation procedure but also reduces the number of unknowns in the LMIs (32)-(34).

## 6 Conservativeness of the Various Stability Analysis Algorithms

The stability criteria presented so far exhibit different levels of conservativeness and complexity in their applications. This is summarized in figure 1. In general, the higher the degree of conservativeness of the

Lyapunov function the lower are the computational requirements to find the corresponding Lyapunov function. The classification in figure 1 has been obtained by considering several examples of piecewise linear systems with two states and cell partitioning crossing the origin. For each field of the diagram we found a system, therefore none of the categories in figure 1 is empty. Some of the relations denoted in figure 1 are quite obvious, for instance it is clear that quadratic stability implies piecewise quadratic stability of the system. The relations become more complex if we consider relaxations and minimal switching sets  $\mathcal{S}$ . For instance systems can be found, that are not piecewise quadratically stable, but are both piecewise quadratically stable with relaxations and  $\mathcal{S}$ -piecewise quadratically stable.



**Figure 1:** Conservativeness of the different stability analysis approaches.

## 7 An Example

To illustrate the analysis and synthesis methods described in the previous sections we consider the following PWA system:

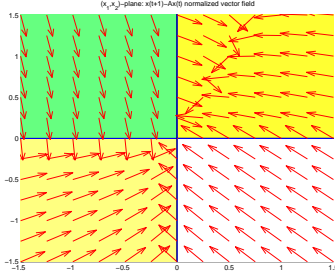
$$x_{k+1} = \begin{cases} \begin{pmatrix} -0.04 & -0.461 \\ -0.139 & 0.341 \end{pmatrix} x_k + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u_k & E_1 x_k \geq 0 \\ \begin{pmatrix} 0.936 & 0.323 \\ 0.788 & -0.049 \end{pmatrix} x_k + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u_k & E_2 x_k \geq 0 \\ \begin{pmatrix} -0.857 & 0.815 \\ 0.491 & 0.62 \end{pmatrix} x_k + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u_k & E_3 x_k \geq 0 \\ \begin{pmatrix} -0.022 & 0.644 \\ 0.758 & 0.271 \end{pmatrix} x_k + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u_k & E_4 x_k \geq 0 \end{cases} \quad (35)$$

The partitioning corresponds to the four quadrants of the two dimensional  $x_1 - x_2$  plane, i.e.

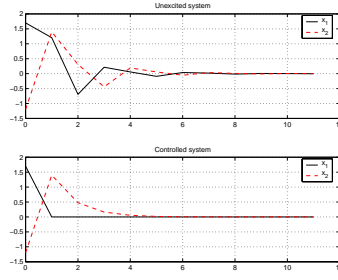
$$E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad E_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad E_3 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \quad E_4 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

For this system the stability analysis gave conclusive results only if we were looking for a piecewise quadratic Lyapunov function with minimal set  $\mathcal{S}$  and relaxations. All the more conservative tests did not give any results. The vector field of the system in the two-dimensional plane can be seen in figure 2-a. The system trajectories starting from some nonzero initial condition are in figure 2-b.

The controller synthesis succeeded in the search for a piecewise linear state feedback showing stability with a

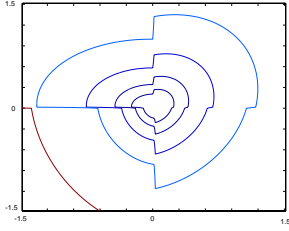


(a)  $(x_1, x_2)$ -plane:  $x(t+1) - Ax(t)$  normalized vector field.

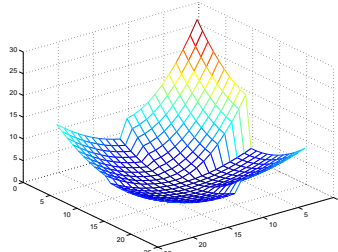


(b) Trajectories of the open loop and the controlled system.

**Figure 2:** Open and closed loop behaviour of system (35).



(a) Contour Plot of the Lyapunov function.



(b) 3D plot of the Lyapunov Function.

**Figure 3:** Open loop Lyapunov function for system (35).

quadratic Lyapunov function. The closed loop matrices are given by the expressions:

$$A_{cli} = A_i + B_i K_i^T \quad (36)$$

where the feedback gains are given by:

$$K_1 = \begin{bmatrix} 0.04 \\ 0.461 \end{bmatrix} \quad K_2 = \begin{bmatrix} -0.936 \\ -0.323 \end{bmatrix}$$

$$K_3 = \begin{bmatrix} 0.857 \\ -0.815 \end{bmatrix} \quad K_4 = \begin{bmatrix} 0.022 \\ -0.644 \end{bmatrix}$$

The contour-lines and a 3-D plot of the Lyapunov function can be seen in figure 3 whereas the state trajectories of the controlled system is depicted in figure 2-b.

## 8 Conclusions

In this paper we presented various algorithms for the stability analysis and the state-feedback design for discrete-time PWA systems. All these procedures hinge on the use of piecewise quadratic Lyapunov functions computed as the solution of a set of LMIs. Due to the LMI formulation, it is important to note that the synthesis criteria we presented can be combined with other design specifications (e.g. robustness requirements) opening the way to multi-objective control for hybrid systems.

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