Null Controllability and Stabilization of Linear Systems Subject to Asymmetric Actuator Saturation [1]

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Abstract

This paper generalizes our recent results on the null controllable regions and the stabilizability of exponentially unstable linear systems subject to symmetric actuator saturation. The description of the null controllable region carries smoothly from the symmetric case to the asymmetric case. As to stabilization, we have to take a quite different approach since the development of our earlier relies mainly on the symmetric property of the vector field. Specifically, in this paper, we construct a Lyapunov function from a closed trajectory to show that this closed trajectory forms the boundary of the domain of attraction for a planar anti-stable system under the control of a saturated linear feedback. If the linear feedback is designed by the LQR method, then there is a unique limit cycle which forms the boundary of the domain of attraction. We further show that if the gain is increased along the direction of the LQR feedback, then the domain of attraction can be made arbitrarily close to the null controllable region. This design can be utilized to construct state feedback laws for higher order systems with two exponentially unstable poles.

1 Introduction

We consider the problem of controlling exponentially unstable linear systems subject to asymmetric actuator saturation. This control problem involves basic issues such as characterization of the null controllable region and stabilizability on the null controllable region. These issues have been focuses of study of and are now well-addressed for linear systems that are not exponentially unstable. For example, it is well-known [2, 8] that such systems are globally null controllable with bounded controls as long as they are controllable in the usual linear system sense.

In regard to stabilizability, it is shown in [9] that a linear system subject to actuator saturation can be globally asymptotically stabilized by smooth feedback if and only if the system is asymptotically null controllable with bounded controls (ANCBC), which, as shown in [2, 8], is equivalent to the system being stabilizable in the usual linear sense and having open loop poles in the closed left-half plane. A nested feedback design technique for designing nonlinear globally asymptotically stabilizing feedback laws was proposed in [11] for a chain of integrators and was fully generalized in [10].

The notion of semi-global asymptotic stabilization on the null controllable region for linear systems subject to actuator saturation was introduced in [5]. The semi-global framework for stabilization requires feedback laws that yield a closed-loop system which has an asymptotically stable equilibrium whose domain of attraction includes an *a priori* given (arbitrarily large) bounded subset of the null controllable region. In [5], it was shown that, for linear ANCBC systems subject to actuator saturation, one can achieve semi-global asymptotic stabilization on the null controllable region using linear feedback laws.

On the other hand, the counterparts of the above mentioned results for exponentially unstable linear systems are less understood. Recently, we made an attempt to systematically study issues related to null controllable regions and the stabilizability on them of exponentially unstable linear systems subject to actuator saturation and gave a rather clear understanding of these issues [3]. Specifically, we gave a simple exact description of the null controllable region for a general anti-stable linear system in terms of a set of extremal trajectories of its time reversed system. We also constructed feedback laws that semi-globally asymptotically stabilize any linear time invariant system with two exponentially unstable poles on its null controllable region. This is in the sense that, for any *a priori* given set in the interior of the null controllable region, there exists a linear feedback law that yields a closed-loop system which has an asymptotically stable equilibrium whose domain of attraction includes the given set. One critical assumption made in [3] is that the actuator saturation is symmetric. The symmetry of the saturation function to a large degree simplifies the analysis of the closed-loop system, it, however, excludes the application of the results to many practical systems.

The goal of this paper is to generalize the results of [3] to the case where the actuator saturation is asymmetric. We take a similar approach as in [3] to characterize

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the null controllable region. In studying the problem of stabilization, we found the methods used in [3] to derive the main results not applicable to the asymmetric case, since the methods rely mainly on the symmetric property of the saturation function. In this paper, we propose a quite different approach to these problems for the asymmetric case.

The proofs are sketched or omitted due to space limitation.

2 Preliminaries and Notation

Consider a linear system

$$\dot{x}(t) = Ax(t) + bu(t), \qquad (1)$$

where $x(t) \in \mathbf{R}^n$ is the state and $u(t) \in \mathbf{R}$ is the control. Given $u^- < 0$ and $u^+ > 0$, let

$$\mathcal{U}_m = \{u : u \text{ is measurable and } u^- \leq u(t) \leq u^+, \forall t \in \mathbf{R}\}.$$

A control signal u is said to be *admissible* if $u \in \mathcal{U}_m$. In this paper, we are interested in the control of the system (1) by using admissible controls. Our first concern is the set of states that can be steered to the origin by an admissible control.

Definition 2.1 *A state x_0 is said to be null controllable if there exist a $T \in [0, \infty)$ and an admissible control u such that the state trajectory $x(t)$ of the system satisfies $x(0) = x_0$ and $x(T) = 0$. The set of all null controllable states is called the null controllable region of the system and is denoted by $\mathcal{C}$.*

With the above definition, we have

$$\mathcal{C} = \bigcup_{T \in [0,\infty)} \left\{ -\int_0^T e^{-A\tau} bu(\tau) d\tau : \ u \in \mathcal{U}_m \right\}. \qquad (2)$$

For simplicity, a linear system and the matrix A are said semi-stable if all the eigenvalues of A are in the closed left half plane; and anti-stable if all the eigenvalues of A are in the open right half plane.

We recall a fundamental result from [2, 6, 8]:

Proposition 2.1 *Assume that (A, b) is controllable.*

a) If A is semi-stable, then $\mathcal{C} = \mathbf{R}^n$.

b) If A is anti-stable, then $\mathcal{C}$ is a bounded convex open set containing the origin.

c) If $A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$ with $A_1 \in \mathbf{R}^{n_1 \times n_1}$ anti-stable and $A_2 \in \mathbf{R}^{n_2 \times n_2}$ semi-stable, and b is partitioned as $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ accordingly, then $\mathcal{C} = \mathcal{C}_1 \times \mathbf{R}^{n_2}$ where $\mathcal{C}_1$ is the null controllable region of the anti-stable system $\dot{x}_1(t) = A_1 x_1 + b_1 u(t)$.

Because of this proposition, we can concentrate on the study of null controllable regions of anti-stable systems. For this kind of systems,

$$\overline{\mathcal{C}} = \left\{ -\int_0^\infty e^{-A\tau} bu(\tau) d\tau : \ u \in \mathcal{U}_m \right\}, \qquad (3)$$

where $\overline{\mathcal{C}}$ denotes the closure of $\mathcal{C}$. We also use "∂" to denote the boundary of a set. In this paper, we will derive a method for explicitly describing $\partial\mathcal{C}$ in Section 3.

In the study of the null controllable regions we will assume, without loss of generality, that (A, b) is controllable and A is anti-stable.

Consider the time reversed system of (1):

$$\dot{z}(t) = -Az(t) - bv(t). \qquad (4)$$

Definition 2.2 *A state z_f is said to be reachable if there exist $T \in [0, \infty)$ and an admissible control v such that the state trajectory $z(t)$ of the system (4) satisfies $z(0) = 0$ and $z(T) = z_f$. The set of all reachable states is called the reachable region of the system (4) and is denoted by $\mathcal{R}$.*

It is known that $\mathcal{C}$ of (1) is the same as $\mathcal{R}$ of (4) (see, e.g., [6]). To avoid confusion, we will continue to use the notation x, u and $\mathcal{C}$ for the original system (1), and z, v and $\mathcal{R}$ for the time-reversed system (4).

3 Null Controllable Regions

In Section 3.1, we show that the boundary of the null controllable region of a general anti-stable linear system with saturating actuator is composed of a set of extremal trajectories of the time reversed system. The descriptions of this set are further simplified for systems with only real poles and for systems with complex poles in Sections 3.2 and 3.3, respectively.

3.1 Description of the null controllable regions

We will characterize the null controllable region $\mathcal{C}$ of the system (1) through studying the reachable region $\mathcal{R}$ of its time reversed system (4).

Since A is anti-stable, we have

$$\begin{aligned} \overline{\mathcal{R}} &= \left\{ -\int_0^\infty e^{-A\tau} bv(\tau) d\tau : \ v \in \mathcal{U}_m \right\} \\ &= \left\{ -\int_{-\infty}^0 e^{A\tau} bv(\tau) d\tau : \ v \in \mathcal{U}_m \right\}. \end{aligned}$$

Noticing that $e^{A\tau} = e^{-A(0-\tau)}$, we see that a point z in $\overline{\mathcal{R}}$ is a state of the time-reversed system (4) at $t = 0$ by applying an admissible control v from $-\infty$ to 0.

Define the asymmetric sign function $\operatorname{sgn}_a(\cdot)$ as

$$\operatorname{sgn}_a(r) := \begin{cases} u^+ & , \quad r > 0, \\ (u^+ + u^-)/2 & , \quad r = 0, \\ u^- & , \quad r < 0. \end{cases}$$

It can be verified that $\text{sgn}_a(r) = \frac{u^+ + u^-}{2} + \frac{u^+ - u^-}{2}\text{sgn}(r)$, where $\text{sgn}(\cdot)$ is the standard sign function.

Theorem 3.1

$$\partial\mathcal{R} = \left\{ -\int_{-\infty}^{0} e^{A\tau} b \,\text{sgn}_a\left(c' e^{A\tau} b\right) d\tau : \quad c \neq 0 \right\}. \quad (5)$$

$\overline{\mathcal{R}}$ is strictly convex. Moreover, for each $z^* \in \partial\mathcal{R}$, there exists a unique admissible control v^* such that

$$z^* = -\int_{-\infty}^{0} e^{A\tau} b v^*(\tau) d\tau. \quad (6)$$

From Theorem 3.1, we see that if v is an admissible control and there is no c such that $v(t) = \text{sgn}_a(c' e^{At} b)$ for $t \leq 0$, then

$$-\int_{-\infty}^{0} e^{A\tau} b v(\tau) d\tau \notin \partial\mathcal{R}$$

and must be in the interior of $\mathcal{R}$.

In what follows, we will simplify (5) and describe $\partial\mathcal{R}$ in terms of a set of trajectories of the time-reversed system (4).

Denote

$$\mathcal{E} := \left\{ v(t) = \text{sgn}_a\left(c' e^{At} b\right), t \in \mathbf{R} : \quad c \neq 0 \right\}, \quad (7)$$

and for an admissible control v, denote

$$\Phi(t, v) := -\int_{-\infty}^{t} e^{-A(t-\tau)} b v(\tau) d\tau. \quad (8)$$

Since A is anti-stable, the integral in (8) exists for all $t \in \mathbf{R}$, so $\Phi(t, v)$ is well defined.

If $v(t) = \text{sgn}_a(c' e^{At} b)$, then

$$\begin{aligned} \Phi(t, v) &= -\int_{-\infty}^{t} e^{-A(t-\tau)} b v(\tau) d\tau \\ &= -\int_{-\infty}^{0} e^{A\tau} b \,\text{sgn}_a\left(c' e^{At} e^{A\tau} b\right) d\tau \in \partial\mathcal{R} \end{aligned}$$

for any $t \in \mathbf{R}$, i.e., $\Phi(t, v)$ lies entirely on $\partial\mathcal{R}$. An admissible control v such that $\Phi(t, v)$ lies entirely on $\partial\mathcal{R}$ is said to be *extremal* and such $\Phi(t, v)$ an *extremal trajectory*. From Theorem 3.1, it can be shown that $\mathcal{E}$ is the set of extremal controls.

Definition 3.1 *$v_1, v_2 \in \mathcal{E}$ are said to be equivalent, denoted by $v_1 \sim v_2$, if there exists an $h \in \mathbf{R}$ such that $v_1(t) = v_2(t - h)$ for all $t \in \mathbf{R}$.*

The following theorem shows that $\partial\mathcal{R}$ is covered by a minimal subset of the extremal trajectories.

Theorem 3.2 *Let $\mathcal{E}_m \subset \mathcal{E}$ be such that for every $v \in \mathcal{E}$, there exists a unique $v_1 \in \mathcal{E}_m$ such that $v \sim v_1$. Then*

$$\partial\mathcal{R} = \{\Phi(t, v) : \quad t \in \mathbf{R}, \; v \in \mathcal{E}_m\}. \quad (9)$$

It turns out that for some classes of systems, $\mathcal{E}_m$ can be easily described. For second order systems, $\mathcal{E}_m$ contains only one or two elements, so $\partial\mathcal{R}$ can be covered by no more than two trajectories; and for third order systems, $\mathcal{E}_m$ corresponds to some real intervals. We will see later that for systems of different eigenvalue structures, the descriptions of $\mathcal{E}_m$ can be quite different.

3.2 Systems with only real eigenvalues

It follows from, for example, [6, p. 77], that if A has only real eigenvalues and $c \neq 0$, then $c' e^{At} b$ has at most $n-1$ zeros. This implies that an extremal control can have at most $n-1$ switches. It was shown in [3] that the converse is also true.

Theorem 3.3 *For the system (4), assume that A has only real eigenvalues, then*

a) an extremal control has at most $n-1$ switches;

b) any bang-bang control with $n-1$ or less switches is an extremal control.

It follows from Theorem 3.3 that $\mathcal{E}_m$ can be chosen as the set of bang-bang controls with $n-1$ or less switches and the first switch is at $t = 0$. Denote $z_e^+ = -A^{-1} b u^+$ and $z_e^- = -A^{-1} b u^-$, then we have,

Observation 3.1 *$\partial\mathcal{R} = \partial\mathcal{C}$ is covered by two bunches of trajectories. The first bunch consists of trajectories of (4) when the initial state is z_e^+ and the input is a bang-bang control that starts at $t = 0$ with u^- and has $n-2$ or less switches. The second bunch consists of the trajectories of (4) when the initial state is z_e^- and the input is a bang-bang control that starts at $t = 0$ with u^+ and has $n-2$ or less switches.*

Furthermore, $\partial\mathcal{R}$ can be simply described in terms of the open-loop transition matrix:

$$\partial\mathcal{R} = \left\{ \left[\sum_{i=1}^{n-1} \pm(u^- - u^+)(-1)^i e^{-A(t-t_i)} \right.\right.$$
$$\left.\left. -\,\text{sgn}_a\left(\pm(-1)^n\right) I \right] A^{-1} b : t_1 \leq t_2 \cdots \leq t \leq \infty \right\},$$

with $t_1 = 0$. For second order systems,

$$\begin{aligned} \partial\mathcal{R} \;=\; & \left\{ e^{-At} z_e^+ - \int_0^t e^{-A(t-\tau)} b u^- d\tau : \; t \in [0, \infty] \right\} \\ & \cup \left\{ e^{-At} z_e^- - \int_0^t e^{-A(t-\tau)} b u^+ d\tau : \; t \in [0, \infty] \right\}. \end{aligned}$$

3.3 Systems with complex eigenvalues

For a system with complex eigenvalues, the set $\mathcal{E}_m$ is harder to determine. In what follows, we consider two important cases.

Case 1. $A \in \mathbf{R}^{2\times 2}$ has a pair of complex eigenvalues $\alpha \pm j\beta$, $\alpha, \beta > 0$.

The set of extremal controls is

$$\mathcal{E} = \{v(t) = \mathrm{sgn}_a(\sin(\beta t + \theta)), t \in \mathbf{R} : \quad \theta \in [0, 2\pi)\}.$$

It is easy to see that

$$\mathcal{E}_m = \{v(t) = \mathrm{sgn}_a(\sin(\beta t)), t \in \mathbf{R}\}$$

contains only one element. Denote $T_p = \frac{\pi}{\beta}$, then $e^{-AT_p} = -e^{-\alpha T_p} I$. Let

$$\begin{aligned} z_s^- &= \left(1 - e^{-\alpha T_p}\right)^{-1} \left(-u^- + e^{-\alpha T_p} u^+\right) A^{-1} b, \\ z_s^+ &= \left(1 - e^{-\alpha T_p}\right)^{-1} \left(-u^+ + e^{-\alpha T_p} u^-\right) A^{-1} b. \end{aligned}$$

It can be verified that the extremal trajectory corresponding to $v(t) = \mathrm{sgn}_a(\sin(\beta t))$ is periodic with period $2T_p$, i.e.,

$$\begin{aligned} \partial\mathcal{R} &= \left\{ e^{-At} z_s^- - \int_0^t e^{-A(t-\tau)} b u^+ d\tau \ : \ t \in [0, T_p] \right\} \\ &\quad \cup \left\{ e^{-At} z_s^+ - \int_0^t e^{-A(t-\tau)} b u^- d\tau \ : \ t \in [0, T_p] \right\} \end{aligned}$$

Case 2. $A \in \mathbf{R}^{3\times 3}$ has eigenvalues $\alpha \pm j\beta$ and α_1, with $\alpha, \beta, \alpha_1 > 0$.

a) $\alpha = \alpha_1$. Then similar to Case 1,

$$\mathcal{E}_m = \{v(t) = \mathrm{sgn}_a(k + \sin(\beta t)), t \in \mathbf{R} : \quad k \in [-1, 1]\}.$$

b) $\alpha \neq \alpha_1$. It can be shown that

$$\mathcal{E}_m = \{u^+, u^-\} \cup \{v(t) = \mathrm{sgn}_a(\sin(\beta t))\} \cup \mathcal{E}_{3m}.$$

where

$$\mathcal{E}_{3m} = \left\{ v(t) = \mathrm{sgn}_a \left(\pm e^{(\alpha_1 - \alpha)t} + \sin(\beta t + \theta) \right) : \theta \in [0, 2\pi) \right\}.$$

Plotted in Fig. 1 are some extremal trajectories on $\partial\mathcal{R}$ of the time-reversed system (4) with

$$A = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.8 & -2 \\ 0 & 2 & 0.8 \end{bmatrix}; \quad B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

$u^+ = 1$, and $u^- = -0.5$.

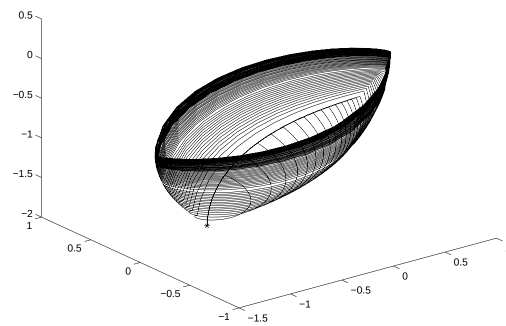

Figure 1: Extremal trajectories on $\partial\mathcal{R}$, $\alpha_1 < \alpha$.

4 Domain of Attraction under Saturated Linear State Feedback

Consider the open loop system

$$\dot{x}(t) = Ax(t) + bu(t) \tag{10}$$

with admissible control $u \in \mathcal{U}_m$. A saturated linear state feedback is given by $u = \mathrm{sat}_a(fx)$, where $f \in \mathbf{R}^{1\times n}$ is the feedback gain and $\mathrm{sat}_a(\cdot)$ is the asymmetric saturation function

$$\mathrm{sat}_a(r) = \begin{cases} u^+, & r > u^+, \\ r, & r \in [u^-, u^+], \\ u^-, & r < u^-. \end{cases}$$

Such a feedback is said to be stabilizing if $A + bf$ is asymptotically stable. With a saturated linear state feedback applied, the closed loop system is

$$\dot{x}(t) = Ax(t) + b\,\mathrm{sat}_a(fx(t)). \tag{11}$$

Denote the state transition map of (11) by $\phi : (t, x_0) \mapsto x(t)$. The domain of attraction $\mathcal{S}$ of the equilibrium $x = 0$ of (11) is defined by

$$\mathcal{S} := \left\{ x_0 \in \mathbf{R}^n : \lim_{t\to\infty} \phi(t, x_0) = 0 \right\}.$$

Obviously, $\mathcal{S}$ must lie within the null controllable region $\mathcal{C}$ of the system (10). Therefore, a design problem is to choose a state feedback gain so that $\mathcal{S}$ is close to $\mathcal{C}$. We refer to this problem as semi-global stabilization on the null controllable region.

We will deal with anti-stable planar systems in this section. Consider the system (11). Assume that $A \in \mathbf{R}^{2\times 2}$ is anti-stable. For the symmetric case where $u^- = -u^+$, it was shown in [3] that $\partial\mathcal{S}$ is the unique limit cycle of the system (11). This limit cycle is unstable for (11) but is stable for the time-reversed system of (11). So it can be easily obtained by simulating the time-reversed system.

However, the method used in [3] to prove the uniqueness of the limit cycle relies on the symmetric property of the vector field. There is no obvious way to generalize the method to the asymmetric case. In this section, we will present a quite different approach to this problem. Actually, we will construct a Lyapunov function from the closed trajectory, and show that the Lyapunov function decreases in time as long as the trajectory starts from within the closed trajectory. Therefore, the open set enclosed by the closed trajectory is the domain of attraction.

Lemma 4.1 *The origin is the unique equilibrium point of the system (11) and there is a closed-trajectory.*

Suppose that Γ is a closed trajectory. By the index theory (e.g., see [4]), Γ must enclose the origin. Also, Γ must have two intersections with one of the lines $fx = u^+$ or $fx = u^-$, or both of them. Otherwise there would be a closed trajectory completely in the linear region of the vector field.

Proposition 4.1 *Denote the region enclosed by the closed trajectory Γ as Ω, then Ω is convex.*

The following theorem shows that under certain condition, a closed trajectory Γ is the boundary of the domain of attraction.

Theorem 4.1 *Let Γ be a closed-trajectory of the system (11). Let the intersections of Γ with the line $\{\mu A^{-1}b : \mu \in \mathbf{R}\}$ be x_{b1} and x_{b2}. If $fx_{b1}, fx_{b2} \in [u^-, u^+]$, i.e., the two intersections x_{b1} and x_{b2} are between the two lines $fx = u^-$ and $fx = u^+$, then $\partial\mathcal{S} = \Gamma$.*

Proof. Denote the region enclosed by Γ as Ω. Since Ω contains the origin in its interior, we can define a Minkowski functional

$$\kappa(x) := \min\{\gamma \geq 0 : x \in \gamma\Omega\}.$$

Clearly, $\kappa(x) = 1$ for all $x \in \Gamma$. Since Γ is a trajectory and the vector field $\dot{x}$ in (11) is continuous, $\partial\kappa(x)/\partial x$ exists and is continuous on Γ. Since Ω is bounded and convex, it follows that $\partial\kappa(x)/\partial x \neq 0$ for all $x \in \Gamma$. Note that $\partial\kappa(x)/\partial x$ is the gradient of the function $\kappa(x)$, so it is perpendicular to the tangent of the curve $\Gamma = \{x \in \mathbf{R}^2 : \kappa(x) = 1\}$, which is $\dot{x}$. Therefore,

$$(\partial\kappa(x)/\partial x)'\, \dot{x} = 0, \quad \forall x \in \Gamma. \tag{12}$$

Define a Lyapunov function as $V(x) := \frac{1}{2}\kappa^2(x)$. It can be verified that for any constant $\alpha > 0$,

$$\kappa(\alpha x) = \alpha\kappa(x), \quad V(\alpha x) = \alpha^2 V(x)$$

and

$$\partial\kappa(x)/\partial x|_{x=\alpha x_0} = \partial\kappa(x)/\partial x|_{x=x_0}.$$

Since $\partial V(x)/\partial x = \kappa(x)\partial\kappa(x)/\partial x$, so $\partial V(x)/\partial x$ exists and is continuous for all $x \in \mathbf{R}^2$. With a detailed investigation of the vector field, it can be shown that for all $x \in \Omega$, along the trajectory of the system (11),

$$\dot{V}(x) = (\partial V(x)/\partial x)'\,(Ax + b\,\mathrm{sat}_a(fx)) \leq 0.$$

It can also be shown that there exists no closed trajectory within Ω. Therefore, all the trajectories starting from within Ω will converge to the origin. It follows that $\partial\mathcal{S} = \partial\Omega = \Gamma$. □

The condition $fx_{b1}, fx_{b2} \in [u^-, u^+]$ in Theorem 4.1 is always true in a special case when the line $\{\mu A^{-1}b : \mu \in \mathbf{R}\}$ is in parallel to the line $fx = u^+$. In the next section, we will show that if f is designed by the LQR method, then the line $\{\mu A^{-1}b : \mu \in \mathbf{R}\}$ is in parallel to the line $fx = u^+$.In this case, any closed-trajectory is the boundary of the domain of attraction and hence there is a unique closed trajectory (a limit cycle). We will further show that the domain of attraction $\mathcal{S}$ can be made arbitrarily close to the null controllable region $\mathcal{C}$ by simply increasing the feedback gain.

5 Semi-Global Stabilization on the Null Controllable Region

We will be focused on second-order anti-stable systems. The result can be easily extended to higher-order systems with two anti-stable mode with the technique in [3]. In this section, we continue to assume that $A \in \mathbf{R}^{2\times 2}$ is anti-stable and (A, b) is controllable.To state the main result of this section, we need to introduce the Hausdorff distance. Let $\mathcal{X}_1, \mathcal{X}_2$ be two bounded subsets of $\mathbf{R}^n$. Then their Hausdorff distance is defined as:

$$d(\mathcal{X}_1, \mathcal{X}_2) := \max\left\{\vec{d}(\mathcal{X}_1, \mathcal{X}_2), \vec{d}(\mathcal{X}_2, \mathcal{X}_1)\right\},$$

where

$$\vec{d}(\mathcal{X}_1, \mathcal{X}_2) = \sup_{x_1 \in \mathcal{X}_1} \inf_{x_2 \in \mathcal{X}_2} \|x_1 - x_2\|.$$

Here the vector norm used is arbitrary.

Let P be the unique positive definite solution of the following Riccati equation,

$$A'P + PA - Pbb'P = 0. \tag{13}$$

Then the origin is a stable equilibrium of the system

$$\dot{x}(t) = Ax(t) + b\,\mathrm{sat}_a(kf_0x(t)) \tag{14}$$

for all $k > 0.5$. Let $\mathcal{S}(k)$ be the domain of attraction of the equilibrium $x = 0$ of (14).

Theorem 5.1 $\lim_{k\to\infty} d(\mathcal{S}(k), \mathcal{C}) = 0$.

Proof. For simplicity and without loss of generality, we assume that

$$A = \begin{bmatrix} 0 & -a_1 \\ 1 & a_2 \end{bmatrix}, \quad a_1, a_2 > 0, \quad b = \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$

Since A is anti-stable and (A, b) is controllable, A, b can always be transformed into this form. With this special form of A and b, we have $f_0 = \begin{bmatrix} 0 & 2a_2 \end{bmatrix}$ and $A^{-1}b = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$. Hence, the line $\{\mu A^{-1}b : \mu \in \mathbf{R}\}$ is actually the line $x_2 = 0$ and it is between the two lines $kf_0x = u^+$ and $kf_0x = u^-$ ($x_2 = \frac{u^+}{2a_2k}$ and $x_2 = \frac{u^-}{2a_2k}$). Therefore, the condition in Theorem 4.1 is satisfied for all $k > 0.5$ and the closed-loop system has a unique limit cycle which is the boundary of $\mathcal{S}(k)$. To visualize the proof, $\partial\mathcal{C}$ and $\partial\mathcal{S}(k)$ for some k, are plotted in Fig. 2, where the inner closed curve is $\partial\mathcal{S}(k) = \Gamma$, and the outer dashed one is $\partial\mathcal{C}$.

For convenience, we proceed the proof with the time reversed system of (14),

$$\dot{z}(t) = -Az(t) - b\,\mathrm{sat}_a(kf_0 z(t)). \qquad (15)$$

Observe that Γ is also the unique limit cycle of this system.

Recall that $\partial\mathcal{C}$ is formed by the trajectories of the system $\dot{z} = -Az - bv$: one from z_e^+(or z_s^+) to z_e^-(or z_s^-) under the control $v = u^-$ and the other from z_e^- (or z_s^-) to z_e^+ (or z_s^+) under the control $v = u^+$. On the other hand, when k is sufficiently large, the limit cycle must have two intersections with each of the lines $kf_0 z = u^+$ and $kf_0 z = u^-$. Suppose that the trajectory starts at the righthand side intersection with $kf_0 z = u^-$, goes clockwise and intersects the two lines successively at time t_1, t_2, t_3, see the points $z(0), z(t_1), z(t_2)$ and $z(t_3)$ in Fig. 2. We also note that from $z(0)$ to $z(t_1)$, $v = \mathrm{sat}_a(kf_0 z) = u^-$ for the closed-loop system (15) and from $z(t_2)$ to $z(t_3)$, $v = u^+$. By comparing the two closed trajectories Γ and $\partial\mathcal{C}$, we can complete the proof by showing that as $k \to \infty$, $z(0), z(t_3) \to z_e^+$(or z_s^+), and $z(t_1), z(t_2) \to z_e^-$ (or z_s^-). $\square$

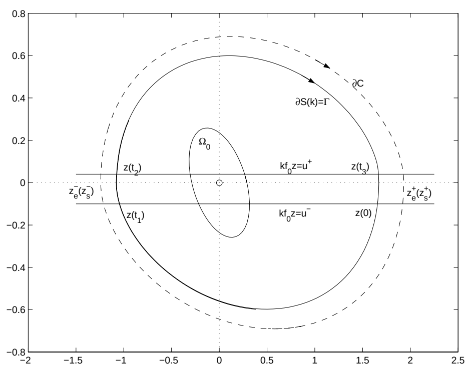

Figure 2: Illustration for the proof of Theorem 5.1

Example 5.1 Consider the open-loop system (1) with $A = \begin{bmatrix} 0.6 & -0.8 \\ 0.8 & 0.6 \end{bmatrix}$, $b = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$, $u^- = -0.5$ and $u^+ = 1$. We have $f_0 = \begin{bmatrix} 0.12 & -0.66 \end{bmatrix}$. In Fig. 3, the boundaries of the domains of attraction corresponding to different $f = kf_0$, $k = 0.50005, 0.6, 0.7, 1, 2$, are plotted from the inner to the outer. It is clear from the figure that the domain of attraction becomes larger as k is increased. The outermost dashed curve is $\partial\mathcal{C}$.

6 Conclusions

In this paper we have studied the problem of controlling a linear system subject to asymmetric actuator saturation. The null controllable region of such a system is first characterized. Simple feedback laws are constructed to stabilize a system with no more than two exponentially unstable open-loop poles. The feedback law guarantees a domain of attraction that includes any given compact set inside the null controllable region.

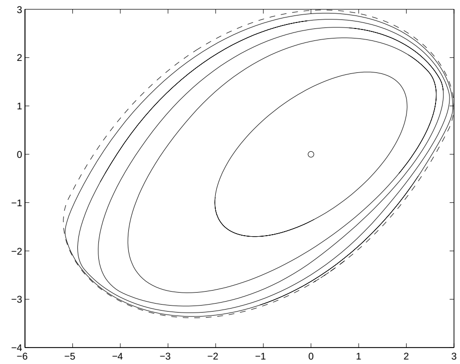

Figure 3: The domains of attraction under different feedback gains

References

[1] J. Alvarez, R. Suarez and J. Alvarez,"Planar linear systems with single saturated feedback," *Systems & Control Letters*, Vol. 20, pp. 319-326,1993.

[2] O. Hájek, *Control Theory in the Plane*, Springer-Verlag, 1991.

[3] T. Hu, Z. Lin and L. Qiu "The controllability and stabilizability of exponentially unstable linear systems with saturating actuators," submitted for publication.

[4] H. K. Khalil, *Nonlinear Systsems*, MacMillan, New York, 1992.

[5] Z. Lin and A. Saberi, "Semi-global exponential stabilization of linear systems subject to 'input saturation' via linear feedbacks," *Systems and Control Letters*, Vol. 21, pp. 225-239, 1993.

[6] J. Macki and M. Strauss, *Introduction to Optimal Control*, Springer-Verlag, 1982.

[7] A. Saberi, Z. Lin and A.R. Teel, "Control of linear systems with saturating actuators," *IEEE Trans. Automat. Contr.* Vol. 41, pp. 368-378, 1996.

[8] E.D. Sontag, "An algebraic approach to bounded controllability of linear systems," *Int. J. Control*, Vol. 39, pp. 181-188, 1984.

[9] E.D. Sontag and H.J. Sussmann, "Nonlinear output feedback design for linear systems with saturating controls," *Proc. 29th IEEE Conf. Decision and Control,* pp. 3414-3416, 1990.

[10] H.J. Sussmann, E.D. Sontag, and Y. Yang, "A general result on the stabilization of linear systems using bounded controls", *IEEE Trans. Automat. Contr.* Vol. 39, pp. 2411-2425, 1994.

[11] A.R. Teel, "Global stabilization and restricted tracking for multiple integrators with bounded controls", *Syst. & Contr. Lett.*, Vol. 18, pp. 165-171, 1992.